Some Ky Fan type inequalities on time scales

1st Cristian Dinu
University of Craiova
Department of Computer Science, Research Center for Artificial Intelligence "Nicolae Ţăndăreanu"
Craiova RO-200585, Romania
c.dinu@yahoo.com

2nd Daniela Dănciulescu
University of Craiova
Department of Computer Science, Research Center for Artificial Intelligence "Nicolae Ţăndăreanu"
Craiova RO-200585, Romania

3rd Alexandru Ţugui
A. I. Cuza University of Iaşi
Iaşi RO-700506, Romania

***Abstract*—In this paper we present some improved versions of the Ky-Fan inequality for functions on time scales, in the framework of some weights that are allowed to take some negative values.**

***Index Terms*—Time scales, convex function, dynamic derivatives, Ky Fan inequality, Jensen inequality**

I. Introduction

The theory of dynamic derivatives on time scales provides an unification and also an extension of traditional differential and difference equations. It is an unification of the discrete theory with the continuous theory, that was introduced by Stefan Hilger in [2]. Using Δ (delta) and ∇ (nabla) dynamic derivatives, a combined dynamic derivative, so called $\Diamond_\alpha$ (diamond-α) dynamic derivative, was introduced as a linear combination of Δ and ∇ dynamic derivatives on time scales. The diamond-α dynamic derivative reduces to the Δ derivative for $\alpha = 1$ and to the ∇ derivative for $\alpha = 0$. Throughout this paper, it is assumed that the basic notions of the time scales calculus are well known and understood. For these, we refer the reader to [1], [2], [9], [11], [12].

The inequality of Ky Fan can be considered a counterpart to the arithmetic-geometric mean inequality. It can be stated as

Theorem 1: If $0 < x_i \leq \frac{1}{2}$, for $i = 1, ..., n$, then

$$\left[\prod_{i=1}^{n} x_i / \prod_{i=1}^{n}(1-x_i)\right]^{\frac{1}{n}} \leq \sum_{i=1}^{n} x_i / \sum_{i=1}^{n}(1-x_i), \tag{1}$$

with equality only if $x_1 = ... = x_n$.

For a given n-tuple of numbers $x = (x_1, ..., x_n)$, the arithmetic, geometric and harmonic means of weight $w = (w_1, ..., w_n)$, (where $w_k \geq 0$ for each k and $\sum_{k=1}^{n} w_k = 1$) are defined as follows

$$\begin{aligned} A_n(x,w) &= \sum_{k=1}^{n} w_k x_k, \\ G_n(x,w) &= \prod_{k=1}^{n} x_k^{w_k}, \\ H_n(x,w) &= \frac{1}{\sum_{k=1}^{n} w_k / x_k}. \end{aligned} \tag{2}$$

It was proved that the inequality (1) works also in the weighted case. Using the above definitions, that is:

$$\frac{G_n(x,w)}{G_n(1-x,w)} \leq \frac{A_n(x,w)}{A_n(1-x,w)} \tag{3}$$

and also the complementary inequality (see [15]),

$$\frac{H_n(x,w)}{H_n(1-x,w)} \leq \frac{G_n(x,w)}{G_n(1-x,w)}. \tag{4}$$

Dragomir and Scarmozzino have improved the above inequalities in [7]. S. Simić gave a converse version of Ky Fan inequality in [14].

A complete weighted version of the Jensen inequality for weights that are allowed to take some negative values was presented by C. Dinu in [5].

Theorem 2: [Theorem 2 in [5]]. Let $a, b \in \mathbb{T}$ and $m, M \in \mathbb{R}$. If $g \in C([a,b]_\mathbb{T}, [m,M])$ and $w \in C([a,b]_\mathbb{T}, \mathbb{R})$ with $\int_a^b w(t)\Diamond_\alpha t > 0$, then the following assertions are equivalent:

(i) w is an α-SP weight for g on $[a,b]_\mathbb{T}$;

(ii) for every $f \in C([m,M],\mathbb{R})$ convex function, we have

$$f\left(\frac{\int_a^b g(t)w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right) \leq \frac{\int_a^b f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}. \tag{5}$$

For the concave functions, the above inequality is reversed. Using this version, we improve the inequalities from [7] and [14].

In section II, we give our main results, regarding the extension of the Ky Fan inequality to a larger class of weights.

II. Extension of the Ky Fan Inequality

For the rest of this paper, let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. We define the weighted means of a function on time scales, that extend the definitions given in (2). For that, we say that a continuous function $w : \mathbb{T} \to \mathbb{R}$ is a α-weight on $[a, b]_{\mathbb{T}}$, provided that $\int_a^b w(t)\Diamond_\alpha t > 0$, where $\alpha \in [0, 1]$.

Definition 3: Let $x : [a, b]_{\mathbb{T}} \to \mathbb{R}_+$ be a continuous positive function and w an α-weight on $[a, b]_{\mathbb{T}}$. We define:

- the generalized weighted arithmetic mean of the function x on the time scale interval $[a, b]$ of weight w:
$$A_{[a,b]}(x, w) = \frac{\int_a^b w(t)x(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}; \quad (6)$$
- the generalized weighted geometric mean of the function x on the time scale interval $[a, b]$ of weight w:
$$G_{[a,b]}(x, w) = \exp\left(\frac{\int_a^b w(t)\ln(x(t))\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right); \quad (7)$$
- the generalized weighted harmonic mean of the function x on the time scale interval $[a, b]$ of weight w:
$$H_{[a,b]}(x, w) = \frac{\int_a^b w(t)\Diamond_\alpha t}{\int_a^b w(t)/x(t)\Diamond_\alpha t}. \quad (8)$$

Example 4:

(i) If $\mathbb{T} = \mathbb{R}$ then, for the α-weight $w : \mathbb{R} \to \mathbb{R}$, (that is $\int_a^b w(t)dt > 0$) we have
$$A_{[a,b]}(x, w) = \frac{\int_a^b w(t)x(t)dt}{\int_a^b w(t)dt};$$
$$G_{[a,b]}(x, w) = \exp\left(\frac{\int_a^b w(t)\ln(x(t))dt}{\int_a^b w(t)dt}\right);$$
$$H_{[a,b]}(x, w) = \frac{\int_a^b w(t)dt}{\int_a^b w(t)/x(t)dt}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$ and $\alpha = 1$, we define $w(i) = w_i$ and $x(i) = x_i$. The condition of 1-weight for w means that $\sum_{i=1}^n w_i > 0$. Then, we have
$$A_{[a,b]}(x, w) = A_n(x, w) = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i};$$
$$G_{[a,b]}(x, w) = G_n(x, w) = \sqrt[w]{\prod_{i=1}^n x_i^{w_i}} \text{ where } w = \sum_{i=1}^n w_i;$$
$$H_{[a,b]}(x, w) = H_n(x, w) = \frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n w_i/x_i}.$$

Remark 5: The generalized means inequality is also true for the generalized weighted means. That is,
$$H_{[a,b]}(x, w) \le G_{[a,b]}(x, w) \le A_{[a,b]}(x, w). \quad (9)$$

For the proof of the right hand of this inequality, we use Theorem 2 for the concave function $f(t) = ln(t)$ and $g = x$. For the left side, we use the same function f and $g = 1/x$.

We can give now our main result.

Theorem 6: Let $x : [a, b]_{\mathbb{T}} \to [m, M]$ be a continuous positive function such that $0 < m \le x(t) \le M \le \frac{\gamma}{2}$, $\gamma > 0$ and w be an α-weight on $[a, b]_{\mathbb{T}}$. Then
$$\frac{A_{[a,b]}(x, w)}{G_{[a,b]}(x, w)} \ge \left(\frac{A_{[a,b]}(x, w)}{G_{[a,b]}(x, w)}\right)^{M^2/(\gamma-M)^2} \ge \frac{A_{[a,b]}(\gamma - x, w)}{G_{[a,b]}(\gamma - x, w)} \ge \left(\frac{A_{[a,b]}(x, w)}{G_{[a,b]}(x, w)}\right)^{m^2/(\gamma-m)^2} \ge 1. \quad (10)$$

Particulary,
$$\frac{A_{[a,b]}(x, w)}{A_{[a,b]}(\gamma - x, w)} \ge \frac{G_{[a,b]}(x, w)}{G_{[a,b]}(\gamma - x, w)}.$$

Proof:

We will adapt an idea used by Dragomir and Scarmozzino in [7] in the context of α-positive weights.

We begin by noticing that $\frac{A_{[a,b]}(x,w)}{G_{[a,b]}(x,w)} \ge 1$, and so, using the fact that $m, M \in (0, \frac{\gamma}{2}]$, we have the first and the last inequality in (10).

Let $f : (0, \gamma) \to \mathbb{R}$ be a function defined by $f(t) = \ln\frac{\gamma-t}{t} + c\ln t$. An easy computation shows that
$$f'(t) = -\frac{\gamma}{t(\gamma - t)} + \frac{c}{t}, \quad t \in (0, \gamma),$$
and
$$f''(t) = \frac{\gamma(\gamma - 2t)}{[t(\gamma - t)]^2} - \frac{c}{t^2} = \frac{1}{t^2}\left[\frac{\gamma(\gamma - 2t)}{(\gamma - t)^2} - c\right], \quad t \in (0, \gamma).$$

Considering the function $g : (0, \gamma) \to \mathbb{R}$, $g(t) = \frac{\gamma(\gamma-2t)}{(\gamma-t)^2}$, we have $g'(t) = \frac{-2\gamma t}{(\gamma-t)^3} < 0$. This shows that the function g is monotonically strictly decreasing on $(0, \gamma)$ and so, for any $m \le t \le M$, we have
$$\frac{\gamma(\gamma - 2M)}{(\gamma - M)^2} = g(M) \le g(t) \le g(m) = \frac{\gamma(\gamma - 2m)}{(\gamma - m)^2}. \quad (11)$$

Using (11), the function f is strictly convex on (m, M) if $c \le \frac{\gamma(\gamma-2M)}{(\gamma-M)^2}$.

Now, we will apply generalized Jensen theorem 2 for the function $f : (m, M) \to \mathbb{R}$, $f(t) = \ln\frac{\gamma-t}{t} + c\ln t$, with $c \le \frac{\gamma(\gamma-2M)}{(\gamma-M)^2}$. We deduce

$$\ln\left(\frac{\gamma-\frac{\int_a^b w(t)x(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}}{\frac{\int_a^b w(t)x(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}}\right)+c\ln\left(\frac{\int_a^b w(t)x(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right)$$
$$=f\left(\frac{\int_a^b w(t)x(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right)\le\frac{\int_a^b w(t)f(x(t))\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}$$
$$=\frac{1}{\int_a^b w(t)\Diamond_\alpha t}\left(\int_a^b w(t)\ln\left(\frac{\gamma-x(t)}{x(t)}\right)\Diamond_\alpha t\right.$$
$$\left.+c\int_a^b w(t)\ln(x(t))\Diamond_\alpha t\right).$$

That is,

$$\ln\left(\frac{A_{[a,b]}(\gamma-x,w)}{A_{[a,b]}(x,w)}\right)+c\ln(A_{[a,b]}(x,w))$$
$$\le\ln\left(\frac{G_{[a,b]}(\gamma-x,w)}{G_{[a,b]}(x,w)}\right)+c\ln(G_{[a,b]}(x,w)),$$

which is equivalent to

$$\ln\left(\frac{G_{[a,b]}(x,w)}{A_{[a,b]}(x,w)}\right)^c\ge\ln\left(\frac{A_{[a,b]}(\gamma-x,w)}{A_{[a,b]}(x,w)}\Big/\frac{G_{[a,b]}(\gamma-x,w)}{G_{[a,b]}(x,w)}\right),$$

or,

$$\left(\frac{G_{[a,b]}(x,w)}{A_{[a,b]}(x,w)}\right)^{c-1}\ge\frac{A_{[a,b]}(\gamma-x,w)}{G_{[a,b]}(\gamma-x,w)}. \tag{12}$$

The best possible choice of c in (12) is its maximal value, that is $c=\frac{\gamma(\gamma-2M)}{(\gamma-M)^2}$, which yields

$$\left(\frac{G_{[a,b]}(x,w)}{A_{[a,b]}(x,w)}\right)^{\frac{\gamma(\gamma-2M)}{(\gamma-M)^2}-1}\ge\frac{A_{[a,b]}(\gamma-x,w)}{G_{[a,b]}(\gamma-x,w)}$$

and now, the second inequality in (10) is obvious.

For the third inequality, we define the function $h(t)=d\ln t-\ln\left(\frac{\gamma-t}{t}\right)$. This function is convex on (m,M) if $d\ge\frac{\gamma(\gamma-2m)}{(1-m)^2}$ and the proof goes in the same manner. ■

The inequality (10) generalizes the main results from [8] and [7].

Remark 7:

(i) If $\mathbb{T}=\mathbb{Z}$, $a=1$, $b=n+1$, $\alpha=1$, defining $w(i)=w_i$ and $x(i)=x_i$ with $\sum_{i=1}^n w_i>0$ and $x_i\in[m,M]\subset(0,\gamma/2]$ for all $i\in\{1,...,n\}$, then, we have

$$\frac{A_n(x,w)}{G_n(x,w)}\ge\left(\frac{A_n(x,w)}{G_n(x,w)}\right)^{M^2/(\gamma-M)^2}\ge\frac{A_n(\gamma-x,w)}{G_n(\gamma-x,w)}$$
$$\ge\left(\frac{A_n(x,w)}{G_n(x,w)}\right)^{m^2/(\gamma-m)^2}\ge 1. \tag{13}$$

(ii) If $\mathbb{T}=\mathbb{R}$ then, for any $a,b\in\mathbb{R}$, any α-weight $w:\mathbb{R}\to\mathbb{R}$, and for any continuous function $x:\mathbb{R}\to\mathbb{R}$, with $x([a,b])\subset[m,M]\subset(0,\gamma/2]$ we have

$$\frac{A_{[a,b]}(x,w)}{G_{[a,b]}(x,w)}\ge\left(\frac{A_{[a,b]}(x,w)}{G_{[a,b]}(x,w)}\right)^{M^2/(\gamma-M)^2}$$
$$\ge\frac{A_{[a,b]}(\gamma-x,w)}{G_{[a,b]}(\gamma-x,w)} \tag{14}$$
$$\ge\left(\frac{A_{[a,b]}(x,w)}{G_{[a,b]}(x,w)}\right)^{m^2/(\gamma-m)^2}\ge 1.$$

The inequalities from (14) represent a continuous version of (13) and an improved continuous version of Theorem 2 from [8].

REFERENCES

[1] Agarwal R.P. and Bohner M., *Basic calculus on time scales and some of its applications*, Results Math. 35 (1999), 3-22

[2] Bohner M. and Peterson A., *Dynamic Equations on Time Scales, An introduction with Applications*, Birkhäuser, Boston, 2001

[3] Czinder P. and Páles Z., *An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means*, J. Inequal. Pure and Appl. Math, **5** (2004), Issue 2, Article no. 42.

[4] Dinu C., *Hermite–Hadamard inequality on time scales*, Journal of Inequalities and Applications, vol. **2008**, Article ID 287947, 24 pages, (2008)

[5] Dinu C., *A weighted Hermite–Hadamard inequality for Steffensen–Popoviciu and Hermite–Hadamard weights on time scales*, Analele Ştiinţifice ale Universităţii "Ovidius", Constanţa, Ser. Mat., **17** (2009)

[6] Dragomir S. S. and Mcandrew A., *Refinements of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure and Appl. Math, **6**(5) (2005), Art. 140.

[7] Dragomir S. S. and Scarmozzino F. P., *On the Ky Fan inequality*, J. Math. Anal. Appl. **269** (2002), 129-136.

[8] Florea A. and Niculescu C. P., *A Note on the Ky Fan Inequality*, J. Ineq. Appl. (2005), Issue 5, 459-468

[9] Hilger S. *Analysis on measure chains – a unified approach to continuous and discrete calculus*, Results Math. **35** (1990), 18-56

[10] Niculescu C. P. and Persson L.-E., *Convex functions and their applications. A contemporary aproach*, Springer-Verlang, Berlin, 2005.

[11] Rogers Jr. J. W. and Sheng Q., *Notes on the diamond-α dynamic derivative on time scales*, J. Math. Anal. Appl., **326** (2007), no. 1, 228-241

[12] Sheng Q,, Fadag M., Henderson M. and Davis J. M., *An exploration of combined dynamic derivatives on time scales and their applications*, Nonlinear Anal. Real World Appl. **7** (2006), no. 3, 395-413

[13] Sidi Ammi M. R., Ferreira R. A. C. and Torres D. F. M., *Diamond-α Jensen's Inequality on Time Scales and Applications*, Journal of Inequalities and Applications, vol. **2008**, Article ID 576876, 13 pages, 2008

[14] Simić S., *On a converse of Ky Fan inequality*, Kragujevac J. Math., **33** (2010), 9599.

[15] Wang W.-L. and Wang P.-F., *A class of inequalities for symmetric functions*, Arta Math. Sinica, **27** (1984), 485-497.