Models for Optimization Decision of Capital And Investment Strategy of Life Insurer with Stochastic Assets and Liabilities

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Abstract — **In this article, we establish three models for determination of optimal capital and investment strategies of life insurance companies based on minimizing total frictional cost without constraint, with constraints of Solvency II and Swiss Solvency Test respectively. We also establish stochastic models of assets and liabilities of insurance companies.**

Keywords— Life Insurer Capital; Reinsurance; Regulation; Solvency

I. INTRODUCTION

In [11] determination of the optimal capital, investment and reinsurance of property and liability insurance companies is modeled, and comparisons are made with different frameworks of capital regulation (e.g., minimizing frictional cost of capital, Solvency II and Swiss Solvency Test). Other relevant works include [1], [2], [4], [5], [6], [7], [8], [9], [12] . The structures of assets and liabilities of life insurance firms is very different from those of property and liability insurers. For example, the mortality, longevity risk, investment risk, and surrender risk are important risks, which must be paid great attention to in the design of risk management strategy for a life insurance form, and for solvency regulation. In particular, rational selection of investment portfolio, as well as capital level, is more important to life insurers because of their characteristic of long term business. [4] presents a discrete time Asset-Liability Management (ALM) model for the simulation of simplified balance sheet of life insurance products. In this paper, we focus on establishing the stochastic model of assets and liabilities of life insurance and determine the optimal investment and capital strategy simultaneously.

II. METHODOLOGY

We now describe our model, which is based on minimizing the total friction cost.

Definition of Variables:

$A(t)$: assets of an insurer at time *t*;

$L(t)$: liabilities of an insurer at time *t*;

X_0 : amount of initial capital;

X_t : amount of capital or surplus at time *t*;

r_c : ratio of frictional capital cost;

$\Pr$: probability of insolvency;

π : amount invested in risky asset;

r : risk-free interest rate;

P_{1t}^{xi} : premium rate of *i*th insurance contract of life insurance at time *t*, $i = 1,2...m_t$;

P_{2t}^{xj} : premium rate of *j*th insurance contract of annuity insurance at time *t*, $i = 1,2...m_t$;

n_t^i : number of *i*th insurance contract in force at time *t*, $i = 1,2...m_t$;

n_{1t}^{xi} : number of life insurance contracts at time *t* with death payment T^{xi} , $i = 1,2...m_1^t$;

n_{2t}^{xi} : number of annuity insurance contracts at time *t*, with survival payment E^{xi} , $i = 1,2...m_2^t$;

σ_1 : volatility of risky asset;

T_t : expected death payments at time *t*;

E_t : expected survival payment at time *t*;

S_t : expected surrender payment at time *t*;

R_t : reserve at time t ;

${}_tq_x$: probability that the insured will die between x and $x+t$ given that he is alive at age x.

${}_tp_x$: probability that the insured survive to age $x+t$ given that he is alive at age x .

s_t^E : expected surrender ratio of annuity insurance at time t ;

s_t^T : expected surrender ratio of life insurance at time t ;

α : ratio of the surrender cost to the reserve;

$R_{t(xi)}^E$: reserve of annuity insurance at time t with survival payment E^{xi} ;

$R_{t(xi)}^T$: reserve of life insurance at time t with death payment T^{xi} ;

$\sigma_{t(xi)}^T$: volatility of death payment of ith life insurance at time t;

$\sigma_{t(xi)}^E$: volatility of survival payment of ith annuity insurance at time t;

$\sigma_{t(xi)}^{ST}$: volatility of surrender payment of ith life insurance at time t;

$\sigma_{t(xi)}^{SE}$: volatility of surrender payment of ith annuity insurance at time t;

v : maximum issue age;

β : the ratio of indirect bankruptcy cost to direct bankruptcy cost.

III. OPTIMIZATION MODEL WITH NO CONSTRAINTS (MODEL 1)

Since the model including reinsurance would become too complicated to be carried out quantitative analysis, we assume that there is no reinsurance. Different from the approaches on the optimal decision of investment and capital level which uses backward dynamic programming, we here use the proper order dynamic programming since we assume that the total frictional cost, $FC(T) \geq 0$, that is, the boundary condition at the end of insurance term is known. Since the difference between the decision in economic and financial environment and engineering is that the former is much more uncertain than the later. It is difficult to estimate the states in all stages exactly in uncertain environment of economy and finance, especially, in the late stages of whole term when the decision term is rather long. Therefore, we believe the order dynamic programming is a better optimal decision method in the uncertain environment. $FC_t,\ t = 0,1,2,....,T$ is defined as the sum the frictional cost of capital, and the expected cost of bankruptcy[12] at time t, that is:

$$FC_t = \sum_{i=1}^{m_1^t} n_{1t}^{xi} P_{1t}^{xi} + \sum_{j=1}^{m_2^t} n_{2t}^{xj} P_{2t}^{xj} + c_c K(t) - c_f E\big(X(t) / X(t) < 0 \big) + C_a \quad (1)$$

where $K(t)$ is the capital needed to be raised or to be reduced at time t, and we refer to it as the adjustment capital required, C_a, where $C_a \geq 0$, is the adjustment cost associated with raising or shedding external capital[3], c_c, is the ratio of frictional capital cost and c_f is the ratio of total bankruptcy cost to firm value (also called the coefficient of bankruptcy cost).

By establishing the objective function of minimizing the sum of the frictional cost of capital, and the expected cost of bankruptcy, we can find the optimal amount of risky asset to invest, the optimal capital level, and therefore the optimal risk-based capital.

The objective function is: Minimize

$$FC_t = \sum_{i=1}^{m_1^t} n_{1t}^{xi} P_{1t}^{xi} + \sum_{j=1}^{m_2^t} n_{2t}^{xj} P_{2t}^{xj} + + c_c K(t) - c_f E\big(X(t) / X(t) < 0 \big) + C_a \quad (2)$$

where $K(t)$ is the capital needed to be raised or to be reduced at time t. When $K(t) > 0$, it means that the insurer raises additional external capital of $K(t)$; otherwise, the insurer reduces capital by $K(t)$ either by paying dividends or repurchasing shares. By solving the objective function for each year, we can determine the optimal capital level, reinsurance, and investment strategy.

IV. STOCHASTIC DIFFERENTIAL EQUATIONS FOR CALCULATING SURPLUS OF LIFE INSURERS

The difference between property-liability insurers and life insurers is that for life insurers, we need to consider different kinds of risks including mortality risk, longevity risk, surrender risk, and investment risk.

Assume that the surplus of life insurance satisfies with the following stochastic differential equation:

[1] Bankruptcy costs can broadly be defined as either direct bankruptcy costs or indirect bankruptcy costs. Direct bankruptcy costs are those explicit costs paid by the debtor in reorganization/liquidation process including legal, accounting, filing and other administrative costs related to the liquidation of the firm's assets. Indirect bankruptcy costs are the opportunity costs of lost management energies [which could lead to] lost sales, lost profits, the higher cost of credit, or possibly the inability of the enterprise to obtain credit or issue securities to finance new opportunities (see [1]).

[2] We assume that there are no costs associated with adjusting to the optimal level of capital.

[3] For multi-period optimization models, it is important to consider the adjustment cost because the adjustment cost will affect the interval of adjusting the capital to the target value (There are significant works on this issue by Leary and Roberts 2005; Flannery and Rangan, 2006; Strebulaev, 2007).

$$dX(t) = d\left(A(t) - L(t)\right) =$$
$$\left(\sum_{i=1}^{m_1^t} n_{1t}^{xi} P_{1t}^{xi} + \sum_{j=1}^{m_2^t} n_{2t}^{xj} P_{2t}^{xj} + \pi X(t)\left(\mu - r\right)\right) dt$$
$$+\left(X(t)r - T_t - E_t - S_t - R_t - X_t r_c\right) dt$$
$$+\pi X(t)\sigma_1 dW_1 + \sum_{x=1}^{v}\sum_{i=1}^{m_1^t} T^{xi}\sqrt{n_{1t}^{xi}}\sigma_{t(xi)}^{T} dW_2^{xi} + \tag{3}$$
$$+\sum_{x=1}^{v}\sum_{j=1}^{m_2^t} E^{xi}\sqrt{n_{2t}^{xi}}\sigma_{t(xj)}^{E} dW_3^{xi} + \sum_{x=1}^{v}\sum_{i=1}^{m_1^t}\sigma_{t(xi)}^{ST} dW_4^{xi} +$$
$$+\sum_{x=1}^{v}\sum_{j=1}^{m_2^t}\sigma_{t(xj)}^{SE} dW_5^{xi}$$

with the boundary condition $X(0) = X_0$ with Model 1, $X(0) = SCR_0$ (referring to Solvency Capital Requirement) with Model 2 and $X(0) = TC_0$ (referring to Target Capital) with Model 3, where W_1, W_2^{xi}, W_3^{xi}, W_4^{xi} and W_5^{xi} are independent Brownian Motions, $m_t = m_1^t + m_2^t$. We have

$$T_t = \sum_{x=1}^{v}\sum_{i=1}^{m_1^t}\left({}_t q_x - {}_{t-1} q_x\right) n_{1t}^{xi}, \tag{4}$$

$$E_t = \sum_{x=1}^{v}\sum_{i=1}^{m_2^t} {}_t p_x n_{2t}^{xi}, \tag{5}$$

$$S_t = \sum_{x=1}^{v}\sum_{i=1}^{m_1^t} {}_t p_x s_t^E R_{t(xi)}^E (1-\alpha) + \sum_{x=1}^{v}\sum_{i=1}^{m_2^t} {}_t p_x s_t^T R_{t(xi)}^T (1-\alpha), \tag{6}$$

and

$$R_t = \sum_{x=1}^{v}\sum_{i=1}^{m_1^t} R_{t(xi)}^E + \sum_{x=1}^{v}\sum_{i=1}^{m_2^t} R_{t(xi)}^T. \tag{7}$$

Based on [7], we let the random variable D_{xt} denote the number of deaths in a population at age x in period between x and $x+t$. Let the random variable L_{xt} denote the number of survivors in a population at age x in period t, and let ω_{xt} be a dummy variable with $\omega_{xt} = 1$, when $e_{xt}^j > 0$ and $\omega_{xt} = 0$, when $e_{xt}^j = 0$, then volatility of mortality is

$$E(D_{xt}) = e_{xt}^j {}_t q_x,$$
$$\sigma_{t(xi)}^T = \frac{\sqrt{Var(D_{xt})}}{\omega_{xt}},$$
$$Var(D_{xt}) = V(E(D_{xt})),$$
$$V(u) = u\left(1 - \frac{u}{e_{xt}^j}\right). \tag{8}$$

where the estimated life expectancy is:

$$\hat{e}_{xt}^j = \frac{\sum_{j>0} L_{xj}(t+j)\left(1 - \frac{1}{2}\hat{q}_{x+j}(t+j)\right)}{L_x(t)}.$$

We also have

$$E(L_{xt}) = e_{xt}^j - e_{xt}^j {}_t q_x,\ \sigma_{t(xi)}^E = \frac{\sqrt{Var(L_{xt})}}{\omega_{xt}},$$
$$Var(L_{xt}) = V(E(L_{xt})),\ V(u) = u\left(1 - \frac{u}{e_{xt}^j}\right) \tag{9}$$

$$R_{t(xi)}^T = \left(R_{t-1(xi)}^T + n_{1t}^{xi} P_{1t}^i\right)\frac{L_{x(t+1)}}{L_{xt}}(1+r) - \frac{D_{xt}}{L_{xt}}, \tag{10}$$

$$R_{t(xi)}^E = \left(R_{t-1(xi)}^E + n_{2t}^{xi} P_{2t}^i\right)\frac{L_{x(t+1)}}{L_{xt}}(1+r) - \frac{L_{x(t+1)}}{L_{xt}}. \tag{11}$$

V. OPTIMIZATION MODEL BASED ON SOLVENCY II (MODEL 2)

We use Value at Risk VaR with $1-\alpha =$ 99.5% of the net asset as the Solvency Capital Requirement (*SCR*) defined by Solvency II. Given confidence level $\alpha \in (0,1)$, the VaR of the net assets at the confidence level $1-\alpha$ is given by the smallest number l such that the probability of the loss X exceeding l is not larger than α. The SCR at time *t* is:

$$SCR_t = VaR_\alpha(\Delta X_t) =$$
$$-\inf\{\Delta X_t \in \Re : P\left(\Delta X_t > l\right) \le \alpha\} = \tag{9}$$
$$-\inf\{l \in \Re : F_{\Delta X_t}(l) \ge \alpha\}$$

where $X(t)$ satisfies stochastic differential equation (3).

In a fashion to the one discussed above, we establish the objective function of minimizing the total frictional cost with the constraint that $\Pr\left(\Delta X_t \le -SCR_t\right) = \alpha$, that is, we minimize

$$FC_t = \sum_{i=1}^{m_1^t} n_{1t}^{xi} P_{1t}^{xi} + \sum_{j=1}^{m_2^t} n_{2t}^{xj} P_{2t}^{xj} +$$
$$c_c K(t) - c_f E\left(X(t) / X(t) < 0\right) + C_a$$

subject to:

$$\Pr\left(\Delta X_t \le -SCR_t\right) = \alpha,\ t = 1,2,...,T.$$

Note that times considered start with t = 1, because the company is assumed to satisfy the regulatory capital requirements at time 0, otherwise it would not be able to continue its existence from that point.

VI. OPTIMIZATION MODEL BASED ON SWISS SOLVENCY TEST (MODEL 3)

Swiss Solvency Test proposes the concept of target capital, which equals the one-year risk capital defined as the expected shortfall of the change of risk-bearing capital during a one-year period. The risk-bearing capital is defined as the difference between the market-consistent value of the assets and the best-estimate of the liabilities.

Based on the definitions above, we establish the formula for calculating the Target Capital (TC) at time *t*:

$$TC_t = ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(\Delta X_t)d\gamma \qquad (10)$$

where X_t satisfies stochastic differential equation (3), $\Delta X_t = X_t - X_{t-1}$ and ES_α is the expected shortfall with confidential level of α . We establish the objective function of minimizing the total frictional cost with the following constraint:

$$TC_t = ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(\Delta X_t)d\gamma, t = 1,2,...,T \ .$$

Note again that times considered start with $t = 1$, because the company is assumed to satisfy the regulatory capital requirements at time 0, otherwise it would not be able to continue its existence from that point.

The model presented seeks to minimize

$$FC_t = $$
$$= \sum_{i=1}^{m_1^t} n_{1t}^{xi} P_{1t}^{xi} + \sum_{j=1}^{m_2^t} n_{2t}^{xj} P_{2t}^{xj} + \qquad (11)$$
$$c_c K(t) - c_f E\left(X(t) / X(t) < 0\right) + C_a$$

subject to: $K(t) \geq TC_t$, where X_t satisfies the appropriate stochastic differential equation and $\Delta X_t = X_t - X_{t-1}$.

VI. CONCLUSION

In this paper, we establish three models of integrated optimization of capital, investment strategies for life insurance companies based on minimizing the total friction cost with no constraint, with the constraint of Solvency II and Swiss Solvency Test respectively. We consider the risks of mortality, investment and surrender, and establish stochastic assets and liabilities models. Further study may focus on the simulation of assets and liabilities of insurance companies and numerical determination of optimal capital and investment strategies and make some comparison among the established three models with examples.

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