Fake p-Values and Mendel Variables: Testing Uniformity and Independence

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***Abstract*—Combining p-values assuming independence and uniformity can be misleading, namely since the "Mendel temptation" to repeat experiments and then to use the most convenient p-value does exist. We discuss Mendel variables, which are mixtures of standard uniform random variables with the minimum or the maximum of two independent standard uniform random variables, that can model the presence of such fake p-values. We also consider variables X that are extremes and products of independent standard uniform random variables in $V = \min(X/Y, (1-X)/(1-Y))$, X and Y independent with support [0,1]. The stability result that X is a Mendel variable implies that V is also a Mendel variable is useful in testing autoregressive serial correlation vs. independence, and also to test non-uniform Mendel vs. uniformity using computationally augmented samples.**

***Index Terms*—uniform random variables, Mendel random variables, fake p-values, combined p-values, independence, uniformity.**

I. Introduction

The classical theory of combined p-values (Pestana, 2011) [12] assumes that those are observations of independent and identically distributed (iid) standard uniform random variables (rvs). But uniformity is solely a consequence of assuming that the null hypothesis is true, and this far-fetched assumption led Tsui and Weerahandi (1989) [15] to introduce the concept of generalized p-values, cf. also Weerahandi (1995) [16], Hung, O'Neill, Bauer and Kohne (1997) [9], and Brilhante (2013) [1].

On the other hand, Pires and Branco (2010) [13] shed some light on the Mendel-Fisher controversy showing that the suspiciously good Mendel results could be explained by assuming that experiments were repeated and only the most convenient result was reported, whenever the result of the original experiment didn't fit what the researcher wished to establish.

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Under the validity of the null hypothesis a p-value is a standard uniform observation, due to the integral transform theorem. Therefore, whenever a researcher repeats the experiment and reports what he considers the most convenient of two observed p values he is in fact recording a *fake p-value* with either a Beta(2,1) or a Beta(1,2) distribution, according to whether he considers more convenient the maximum or the minimum of the two observed values.

The combination of p-values $p_1, p_2, \ldots, p_n$ must therefore consider the possibility that some of those are Beta(2,1) or Beta(1,2) fake p-values. This implies a drastic change in the underlying distribution theory, by considering that the p_k's are observations from random variables P_k that are mixtures X_m of $1-\frac{|m|}{2}$ independent uniform and of $\frac{|m|}{2}$ Beta(2,1) or Beta(1,2) random variables, that we shall call Mendel(m) random variables and denote by $X_m \sim Mendel(m)$. These mixtures have been thoroughly studied in Gomes, Pestana, Sequeira, Mendonça and Velosa (2009) [8] in the context of sample computational augmentation to test uniformity.

In Section 2 we discuss Mendel random variables, $X_m \sim Mendel(m)$, $m \in [-2,2]$, in the context of tilting the standard uniform probability density function (thus no tilting, $m=0$, leaves the uniform distribution unchanged).

Section 3 is devoted to the investigation of extensions of Deng and George (1992) [6] characterization of the standard uniform random variable, using $V = \min\left(\frac{X}{Y}, \frac{1-X}{1-Y}\right)$ when X and Y are independent random variables with support [0,1]. In particular we show that:

- The class of Mendel random variables is closed for the above operation, namely $V = \min\left(\frac{X_m}{X_p}, \frac{1-X_m}{1-X_p}\right) \stackrel{\mathrm{d}}{=} X_{\frac{mp}{6}}$.
- More generally, if X and Y with support [0,1] are

independent, with $X \sim Mendel(m)$,

$$V = \min\left(\frac{X_m}{Y}, \frac{1-X_m}{1-Y}\right) \stackrel{\mathrm{d}}{=} X_{m(2\mathbb{E}[Y]-1)}.$$

Therefore $V \stackrel{\mathrm{d}}{\approx} X$ if $\mathbb{E}[Y] \approx 1$. Observe that Deng and George (1992) [6] characterization of the standard uniform $X \sim Mendel(0)$ shows that in that case $V \sim Uniform(0,1)$ with Y and V independent.

When meta-analyzing p-values, it is of the utmost importance to test independence — we address in particular the investigation of independence vs. autoregressive serial correlation in Section 4 — and uniformity vs. a $Mendel(m),\ m \neq 0$, setup.

The estimation of the proportion $\frac{|m|}{2}$ of fake p-values is quite complex and we don't have so far a full-proof methodology to achieve it, cf. Brilhante, Pestana, Semblano and Sequeira (2015) [4].

II. Tilting the Uniform and Mendel Variables

Let U denote a standard uniform random variable. When tilting the probability density function of U with pole (0.5,1), for $m \in [-2,2]$, we obtain a probability density function of a variable X_m given by

$$f_m(x) = \Big(mx + 1 - \frac{m}{2}\Big)\,\mathbb{I}_{(0,1)}(x). \qquad (1)$$

We shall say that $X_m \sim Mendel(m)$. It is obvious that $X_0 \sim Uniform(0,1)$, $X_{-2} \sim Beta(1,2)$ is the minimum of two independent standard uniform rvs, and $X_2 \sim Beta(2,1)$ is the maximum of two independent standard uniform rvs.

For intermediate values of $m \in (-2,0)$, X_m is a mixture of a standard uniform, with weight $1-\frac{|m|}{2}$, and Beta(1,2) rvs, and for $m \in (0,2)$ it is a mixture of standard uniform and Beta(2,1) rvs:

$$X_m = \left\{\begin{array}{cc} U & U_{i:2} \\ \\ 1-\frac{|m|}{2} & \frac{|m|}{2} \end{array}\right., \quad i=1,2, \qquad (2)$$

where $i=1$ if $m \in [-2,0]$ and $i=2$ if $m \in (0,2]$, and $U_{1:2}$ and $U_{2:2}$ denote, respectively, the minimum and maximum of two independent standard uniform random variables.

Gomes, Pestana, Sequeira, Mendonça and Velosa (2009) [8] used this family of variables in the context of testing uniformity using augmented samples, and the reason to call them Mendel random variables stems out from the interesting explanation devised by Pires and Branco (2010) [13] for the outstanding performance of Mendel experiments, that Fisher accused of being too good to be true. In fact, a possible explanation is the repetition of experiments whose results weren't in accordance with Mendel theory, reporting the "best" of the two p-values obtained.

Observe also that $X_m \sim Mendel(|m|)$ is $N(p)$-max-infinitely divisible, with $N(p) = \left\{\begin{array}{cc} 1 & 2 \\ p & 1-p \end{array}\right.$, and $p = 1-\frac{|m|}{2} \in [0,1]$, since X_m may be interpreted as a random maximum with $N(p)$ subordinator (cf., e.g., the work by Mendonça, Pestana and Ivette (2015) [11]). In fact, considering a sequence of iid rvs $\{W_n\}_{n\in\mathbf{N}}$, identically distributed to a standard uniform rv W and independent from $N(p)$ we have

$$X_m \stackrel{d}{=} W_{N(p):N(p)|N(p)\geq 1}.$$

III. On $\min\left(\frac{X}{Y}, \frac{1-X}{1-Y}\right)$ when X and Y are Independent Beta or Mendel Variables

Let X and Y be independent random variables with support $\mathcal{S} = [0,1]$, and define

$$V = \min\left(\frac{X}{Y}, \frac{1-X}{1-Y}\right). \qquad (3)$$

Deng and George (1992) [6] established an useful characterization of the standard uniform distribution using the random variable V in (3):

$$X \sim Uniform(0,1) \Longleftrightarrow V \sim Uniform(0,1) \qquad (4)$$

with Y, V independent.

Observe that $X \sim Uniform(0,1)$ is the $Mendel(0)$ random variable, which is a $Beta(1,1)$ random variable, or a $BetaBoop(1,1,1,1)$ of the more general $BetaBoop(p,q,\pi,\rho)$ family of random variables with probability density function given by

$$f_{p,q,\pi,\rho}(x) = c_{p,q,\pi,\rho}\, x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{\pi-1}(-\ln x)^{\rho-1}\mathbb{I}_{(0,1)}(x),$$

$c_{p,q,\pi,\rho} = \int_0^1 x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{\pi-1}(-\ln x)^{\rho-1}\mathrm{d}x$ and $p,q,\pi,\rho > 0$, introduced by Brilhante, Gomes and Pestana (2011) [3]. Further observe that the $BetaBoop(1,1,1,n)$ random variable is the product of n independent standard uniform random variables.

In what follows we investigate the distribution of the random variables V in (3) when X is a Mendel random variable, an extreme of a sequence of independent standard uniform rvs, or the product of independent uniform rvs.

The distribution function of V is

$$\begin{aligned} F_V(z) &= \mathbb{P}(X \leq Yz) + \mathbb{P}(X \geq 1-(1-Y)z) \\ &= 1 - \int_0^1 \left[F_X(1-(1-y)z) - F_X(yz)\right] f_Y(y)\mathrm{d}y \end{aligned}$$

and its probability density function in the support [0,1] is

$$\int_0^1 \left[y f_X(yz) + (1-y) f_X(1-z+zy)\right] f_Y(y)\mathrm{d}y, \qquad (5)$$

where F_X denotes the distribution function of X and f_X and f_Y are the probability density functions of X and Y, respectively.

(a) If X is the maximum of two independent standard uniform rvs and Y is the product of two independent standard uniform rvs, X and Y independent, then $f_V(z) = \left(\frac{3}{2} - z\right)\mathbb{I}_{(0,1)}(z)$.

More generally, if X is the maximum of two independent standard uniform rvs and Y with support [0,1] has expectation $\mathbb{E}[Y]$, with X and Y independent, the probability density function of V is

$$f_V(z) = [(4\mathbb{E}[Y]-2)z + 2(1-\mathbb{E}[Y])]\,\mathbb{I}_{(0,1)}(z). \quad (6)$$

For instance, if Y is the product of n independent standard uniform rvs, then $V \sim Mendel\left(\frac{1-2^{n-1}}{2^{n-2}}\right)$. If $Y \sim Beta(p,q)$, $V \sim Mendel\left(2\,\frac{p-q}{p+q}\right)$, and in particular

- if Y is the maximum of n independent standard uniform rvs, then $V \sim Mendel\left(\frac{2(n-1)}{n+1}\right)$,
- if Y is the minimum of n independent standard uniform rvs, then $V \sim Mendel\left(\frac{2(1-n)}{n+1}\right)$,
- more generally if Y is the k-th ascending order statistic in a sequence of n independent standard uniform rvs, then $V \sim Mendel\left(\frac{4k-2n-2}{n+1}\right)$.

Also, observe that if the expectation of Y is $\frac{1}{2}$, then $V \sim Uniform(0,1)$.
Observe however that this doesn't contradict Deng and George characterization of the uniform, since in this (4) case V is not independent of Y.
In this context, it seems worthwhile to quote from Johnson, Kotz and Balakrishnan (1995, p. 286) [10]:

> "*These results provide a partial answer to the important problem of determining the family of functions g for which the uniformity of U and V implies* [...] *uniformity of g(U,V) if U and V are independent random variables having support* (0,1). (*This is relevant to construct methods for improving pseudorandom number generators to make them give results closer to standard uniform distributions.*)"

cf. also Gomes, Pestana, Sequeira, Mendonça and Velosa (2009).
Similarly, if X is the minimum of two independent standard uniform rvs and $Y \sim Beta(p,q)$, $f_V(z) = \frac{2}{p+q}\left[p + (q-p)\,z\right]\mathbb{I}_{(0,1)}(z)$.

(b) The above results are particular cases obtained when $X \sim Mendel(m)$.

Theorem:
If X and Y are independent random variables with $X \sim Mendel(m)$, then

$$V = \min\left(\frac{X}{Y}, \frac{1-X}{1-Y}\right) \sim Mendel\left((2\mathbb{E}[Y]-1)m\right).$$

Proof:
As $f_X(x) = \left(mx + 1 - \frac{m}{2}\right)\mathbb{I}_{(0,1)}(x)$, from

$$f_V(z) = \int_0^1 f_Y(y)\left[y\left(mzy + 1 - \frac{m}{2}\right) + \right.$$
$$\left. + \,(1-y)\left(m - mz(1-y) + 1 - \frac{m}{2}\right)\right] dy$$

for $z \in (0,1)$, we obtain

$$f_V(z) = (2\mathbb{E}[Y]-1)mz + 1 - \frac{m(2\mathbb{E}[Y]-1)}{2}.$$

□

Corollary:
Let X and Y be independent random variables, $X \sim Mendel(m)$ and $Y \sim Mendel(p)$. Then

$$V = \min\left(\frac{X}{Y}, \frac{1-X}{1-Y}\right) \sim Mendel\left(\tfrac{mp}{6}\right). \quad (7)$$

Proof:
As $\mathbb{E}[Y] = \frac{1}{2} + \frac{p}{12}$, it follows that the Mendel parameter is $(2\mathbb{E}[Y]-1)m = \frac{mp}{6}$.

□

Observe that, as for $m,p \in [-2,2]$ we have $1 - \frac{|mp|}{12} \geq \max\left(1 - \frac{|m|}{2}, 1 - \frac{|p|}{2}\right)$, it follows that the uniform component of $V \stackrel{d}{=} X_{\frac{mp}{6}}$ weights more than the uniform component of either X_m or X_p.

(c) If $X \sim Beta(3,1)$ and $Y \sim Beta(n,1)$

$$f_V(z) = \frac{3}{n+1}\left[1 - \frac{4z}{n+2} + \frac{(n^2+2)z^2}{n+2}\right]\mathbb{I}_{(0,1)}(z).$$

In particular, if $Y \sim Uniform(0,1)$ then $f_V(z) = \left(\frac{3}{2} - 2z + \frac{3z^2}{2}\right)\mathbb{I}_{(0,1)}(z)$; and if Y is the maximum of two independent standard uniform rvs then $f_V(z) = \left(1 - z + \frac{3z^2}{2}\right)\mathbb{I}_{(0,1)}(z)$.

(d) If $X \sim Beta(2,2)$ and $Y \sim Beta(n,1)$, V has probability density function

$$6\left(\left(1 - \frac{2n}{n+1} + \frac{2n}{n+2}\right)z + \left(\frac{3n}{n+1} - \frac{3n}{n+2} - 1\right)z^2\right)$$

in the support [0,1].
For the simple cases $Y \sim Uniform(0,1)$ and $Y \sim Beta(2,1)$ — i.e., the maximum of two independent uniform rvs — we get the same result, $f_V(z) = \left(4z - 3z^2\right)\mathbb{I}_{(0,1)}(z)$.

(e) More generally, if $X \sim Beta(p,q)$ we get, $f_V(z) = \int_0^1 A(z,y) f_Y(y)\mathrm{d}y$ for $z \in (0,1)$, where $A(z,y) =$

$$\frac{y^p z^{p-1}(1-zy)^{q-1} + [1-z(1-y)]^{p-1}(1-y)^q z^{q-1}}{B(p,q)}.$$

Therefore

$$f_V(z) = \frac{1}{B(p,q)}\left[\sum_{r=0}^{q-1}\binom{q-1}{r}(-1)^r z^{p+r-1}\mathbb{E}[Y^{p+r}]\right.$$
$$\left. + \sum_{s=0}^{p-1}\binom{p-1}{s}(-1)^s z^{q+s-1}\mathbb{E}[(1-Y)^{s+q}]\right]$$

and in particular if $p = q$ we get $f_V(z) =$

$$\frac{z^p}{B(p,q)}\sum_{r=0}^{p-1}\binom{p-1}{r}(-1)^r z^{r-1}\mathbb{E}\left[Y^{p+r} + (1-Y)^{p+r}\right].$$

For $p=q=2$, $f_V(z)=$

$$6\left[2z\left(\mathbb{E}[Y^2-Y+\frac{1}{2}\right)-3z^2\left(\mathbb{E}[Y^2-Y+\frac{1}{3}\right)\right],$$

of which examples from (d) are special cases.
If $X \stackrel{\mathrm{d}}{=} Y \sim Beta(2,2)$, $\mathbb{E}[Y]=\frac{1}{2}$ and $\mathbb{E}[Y^2]=\frac{3}{10}$, and therefore $f_V(z)=\left(\frac{6}{5}\,z\,(3-2z)\right)\mathbb{I}_{(0,1)}(z)$.

For $p=q=3$,

$$f_V(z)=\frac{z^3}{B(3,3)}\left\{\frac{\mathbb{E}[Y^3+(1-Y)^3]}{z}\right.$$
$$\left.-2\mathbb{E}\left[Y^4+(1-Y)^4\right]+z\left[\mathbb{E}[Y^5+(1-Y)^5]\right]\right\}.$$

If $Y \sim Beta(\alpha,\alpha)$ as $\mathbb{E}[Y^k]=\mathbb{E}[(1-Y)^k]$ the above expression is very easy to compute. For instance, if $Y \sim Beta(2,2)$

$$f_V(z)=\left(12z^2-\frac{120}{7}z^3+\frac{45}{7}z^4\right)\mathbb{I}_{(0,1)}(z).$$

(f) The probability density function of V may also be computed conditioning on the value of X:

$$f_V(z)=\frac{1}{z^2}\int_0^1 f_X(x)$$
$$\left[xf_Y\left(\frac{x}{z}\right)-(x-1)f_Y\left(\frac{z+x-1}{z}\right)\right]\mathrm{d}x.$$

So, if $Y \sim Mendel(m)$,

$$f_V(z)=\frac{1}{z^2}\left\{\frac{m}{z}\int_0^z x^2\left[f_X(x)-f_X(1-x)\right]\mathrm{d}x\right.$$
$$+\left(1-\frac{m}{2}\right)\int_0^z x\left[f_X(x)+f_X(1-x)\right]\mathrm{d}x$$
$$\left.+m\int_0^z xf_X(1-x)\mathrm{d}x\right\}.$$

Thus if $f_X(x)=f_X(1-x)$ and $Y \sim Mendel(m)$ the density of V doesn't depend on the Mendel parameter:

$$f_V(z)=\frac{\int_0^z 2x\,f_X(x)\mathrm{d}x}{z^2}.$$

For instance, if $X \sim Beta\left(\frac{1}{2},\frac{1}{2}\right)$, $f_V(z)=$

$$\left(\frac{1}{z^2}+\frac{\sqrt{z-1}\,\mathrm{arcsinh}\sqrt{z-1}-\sqrt{z(1-z)}}{\pi z^2}\right)\mathbb{I}_{(0,1)}(z),$$

and if $X \sim Beta(2,2)$

$$f_V(z)=\left(4z-3z^2\right)\mathbb{I}_{(0,1)}(z).$$

(g) If $X \stackrel{\mathrm{d}}{=} 1-X$ and $Y \stackrel{\mathrm{d}}{=} 1-Y$ the expression (5) may be simplified:

$$f_V(z)=2\int_0^1 f_X(yz)f_Y(y)\mathrm{d}y.$$

For instance:

- If $X \sim Beta(3,3)$ and $Y \sim Uniform(0,1)$,
$$f_V(z)=\left(15z^2-24z^3+10z^4\right)\mathbb{I}_{(0,1)}(z).$$
- If $X \stackrel{d}{=} Y \sim Beta(3,3)$,
$$f_V(z)=\left(\frac{75}{7}z^2-\frac{100}{7}z^3+5z^4\right)\mathbb{I}_{(0,1)}(z).$$

IV. Testing the Independence of $\boldsymbol{p}$-values

Testing independent standard uniform rvs vs. autoregressive Mendel is relevant in the context of meta-analyzing p-values.

Let $\{X_{m,i}\}$, $i \geq 0$ be a sequence of replicas of independent Mendel variables X_m, $m \in [-2,2]$. Define

$$Y_{m,i}=\rho\,Y_{m,i-1}+(1-\rho)\,X_{m,i},\quad Y_{m,0}=X_{m,0},$$

$1 \leq i \leq n$, $\rho \in [0,1)$.

If $\rho=0$, the sequence $\{Y_{m,i}\}$, $i \geq 0$, is the initial one. But if $\rho>0$ there is serial correlation.
The inverse transformation, $X_{m,i}=\frac{Y_{m,i}-\rho\,Y_{m,i-1}}{1-\rho}$, with $1 \leq i \leq n$, and $J=\left(\frac{1}{1-\rho}\right)^n$, leads to,

$$f_{Y_{m,1},\ldots,Y_{m,n}}(y)=\prod_{i=1}^n\left(m\,\frac{y_i-\rho\,y_{i-1}}{1-\rho}+\frac{2-m}{2}\right)J\,\mathbb{I}_{\boldsymbol{S}}(y),$$

where $(y)=(y_1,...,y_n)\in[0,1]^n$ and

$$\boldsymbol{S}=\bigcap_{i=1}^n\left\{(y_1,\ldots,y_n)\in[0,1]^n : 0<\frac{y_i-\rho\,y_{i-1}}{1-\rho}<1\right\}.$$

As $\forall i \in \{1,\ldots,n\}, 0<\frac{y_i-\rho\,y_{i-1}}{1-\rho}<1$ is equivalent to

$$\rho<\min_{1\leq i\leq n}\min\left\{\frac{y_i}{y_{i-1}},\frac{1-y_i}{1-y_{i-1}}\right\}=: A(y),$$

it follows that in the case $m=0$ the joint density of $Y_1,\ldots,Y_n$ is

$$f_{Y_1,\ldots,Y_n}(y)=J\,\mathbb{I}_{\{(y)\in[0,1]^n:\,\rho<A(y)\}}(y),$$

and we have to solve

$$\min_{1\leq i\leq n}\min\left\{\frac{X_{0,i}}{X_{0,i-1}},\frac{1-X_{0,i}}{1-X_{0,i-1}}\right\}=\min_{1\leq i\leq n}\{U_1,\ldots,U_n\},$$

where $\{U_1,\ldots,U_n\}$ is a sequence of independent standard uniform random variables, and therefore $\min_{1\leq i\leq n}\{U_1,\ldots,U_n\}\sim Beta(1,n)$.

More generally assuming $\rho>0$, $m=0$,

$$\min\left\{\frac{Y_{0,i}}{Y_{0,i-1}},\frac{1-Y_{0,i}}{1-Y_{0,i-1}}\right\}=$$
$$\min\left\{\rho+(1-\rho)\frac{X_{0,i}}{Y_{0,i-1}},\rho+(1-\rho)\frac{1-X_{0,i}}{1-Y_{0,i-1}}\right\}$$

is uniform with support $(\rho,1]$; denote

$$\min\left\{\rho+(1-\rho)\frac{X_{0,i}}{Y_{0,i-1}},\rho+(1-\rho)\frac{1-X_{0,i}}{1-Y_{0,i-1}}\right\}=V_{i,\rho},$$

$V = \min_{1\leq i\leq n} V_{i,\rho}$. V is the ML estimator of ρ, sufficient for ρ. The likelihood function is $L(\rho) = \left(\frac{1}{1-\rho}\right)^n \mathbb{I}_{\rho\leq V}$.

Therefore, reject independence if $V > 1-\alpha^{1/n}$, the power being

$$\begin{cases} \dfrac{\alpha}{(1-\rho)^n} & \text{if } \ \rho \leq 1-\alpha^{1/n} \\ \\ 1 & \text{otherwise} \end{cases} .$$

Finally, for a general $m \in [-2, 2]$,

$$\frac{Y_{m,i}}{Y_{m,i-1}} = \rho + (1-\rho)\,\frac{X_{m,i}}{Y_{m,i-1}}$$

and

$$\frac{1-Y_{m,i}}{1-Y_{m,i-1}} = \rho + (1-\rho)\,\frac{1-X_{m,i}}{1-Y_{m,i-1}}$$

implying that

$$\min\left(\frac{Y_{m,i}}{Y_{m,i-1}}, \frac{1-Y_{m,i}}{1-Y_{m,i-1}}\right) =$$

$$\rho + (1-\rho)\,\min\left(\frac{X_{m,i}}{Y_{m,i-1}}, \frac{1-X_{m,i}}{1-Y_{m,i-1}}\right)$$

and from the independence of the $X_{m,i}$'s from the Corollary in Section 3 we get

$$\min\left(\frac{Y_{m,i}}{Y_{m,i-1}}, \frac{1-Y_{m,i}}{1-Y_{m,i-1}}\right) \stackrel{\mathrm{d}}{=} \rho + (1-\rho)\, X_{\frac{m^2}{6}}\,.$$

V. Testing the Uniformity vs. Mendel(m) Distribution of p-values

Let $p_1, p_2, \ldots, p_k$ be a sequence of p-values obtained when testing some null hypothesis H_0 in independent experiments; the rationale for combining p-values under the validity of the null hypothesis is well-established, for instance Fisher (1932) [7] used $-2\sum_{i=1}^{k} \ln P_i \sim \chi^2_{2k}$ and Tippett (1931) [14] used $\min\limits_{1\leq i\leq k} \{P_i\} \sim Beta(1,k)$ to test the overall validity of H_0.

But under the validity of H_0 may we assume that they are observations from $P_i \sim Uniform(0,1)$, or is it possible that some of those recorded p_k's are fake p-values?

Maintaining uniformity using standard tests may be a weak decision, resulting from the fact that there exist very few p_i's.

We can however compute

$$v_i = \min\left(\frac{p_i}{b_i}, \frac{1-p_i}{1-b_i}\right),\ i = 1, ..., k, \tag{8}$$

using for instance Beta(2,1) — and thus Mendel(2) — pseudo random numbers b_i, quite easy to generate.

If the P_i's are uniform, V_i will also be uniform and independent of the initial set of p_i's, otherwise if the P_i's are $Mendel(m)$ the V_i's will be Mendel($\frac{m}{3}$). Either way, we shall now have an augmented set $\{p_1, ..., p_k, v_1, ..., v_k\}$ to test uniformity.

This procedure may indeed be repeated to have an augmented set of size $3k$, and then of size $4k$, and so on. Observe however that the Mendel parameters decay from the original m to $\frac{m}{3}$, to $\frac{m}{9}$, to $\frac{m}{27}$, and so on, and thus the generated values will be from models closer and closer to the standard uniform, cf. Brilhante, Mendonça, Pestana and Sequeira (2010) [2] and Brilhante, Pestana and Sequeira (2010) [5], and therefore with very tiny contribution to collect evidence leading to rejection of the uniformity null hypothesis.

Thus this apparently appealing recursive procedure based on the Corollary presented in Section 3 can decrease drastically the power of the test, and must be used sparingly. In fact, using Mendel pseudo-random numbers b_i in the denominator of (8) and the corollary in section 3 to artificially increase the sample size will fatally decrease the power of the test, an apparently awkward result in Gomes, Pestana, Sequeira, Mendonça and Velosa (2009) [8] when these authors considered in their simulations Y a Mendel random variable.

On the other hand, the Theorem in Section 3 opens new possibilities, since we are no longer limited to use a Mendel variable in the denominator. Indeed, if we compute

$$v_i = \min\left(\frac{p_i}{y_i}, \frac{1-p_i}{1-y_i}\right),,\ i = 1, ..., k,$$

where the pseudo-random numbers y_i are from Y with a chosen $\mathbb{E}[Y] \approx 1$, the v_k will be from a $V \sim Mendel\left((2\mathbb{E}[Y]-1)m\right)$ as close to the $Mendel(m)$ as we wish, even though the parameter m is unknown and being subject to testing.

We performed an elementary Monte Carlo study to compare the power of the Kolmogorov-Smirnov's goodness of fit test for the null hypothesis of uniformity for a sample $(p_1, \ldots, p_k)$ and for the computationally augmented sample $(p_1, \ldots, p_k, v_1, \ldots, v_k)$, where the p_i's are observations from a $X \sim Mendel(m)$ random variable and the y_i's, used to obtain the v_i's from the random variable V, are observations from a random variable $Y \stackrel{d}{=} 1 - \prod_{i=1}^{10} U_i$, with $U_1, \ldots, U_{10}$ independent standard uniform random variables. Observe that $\mathbb{E}(Y) = 1 - \left(\frac{1}{2}\right)^{10} \approx 1$, and thus the Mendel's parameter of V is approximately equal to m, the value of the parameter of X, which will not be known in pratice.

In Fig. 1 we show the results for the proportion of rejections of uniformity, for the significance level 0.05. As we can observe, the power of the test does increase as we augment the sample from a size k to a size $2k$, where $k = 5, 10, 15, 20$.

These results show that if there is some evidence against uniformity in the initial sample $(p_1, \ldots, p_k)$, this will also happen in the augmented sample, with the power of the test being higher for the larger samples, as desired.

Further, we have done the same type of study for the Fisher $-2\sum_{i=1}^{k} \ln P_i \sim \chi^2_{2k}$ test on the combined p-values to decide on the overall null hypothesis. The results are shown in Fig. 2, and as we can observe, we have the same type of conclusions as those for the Kolmogorov-Smirnov's test.

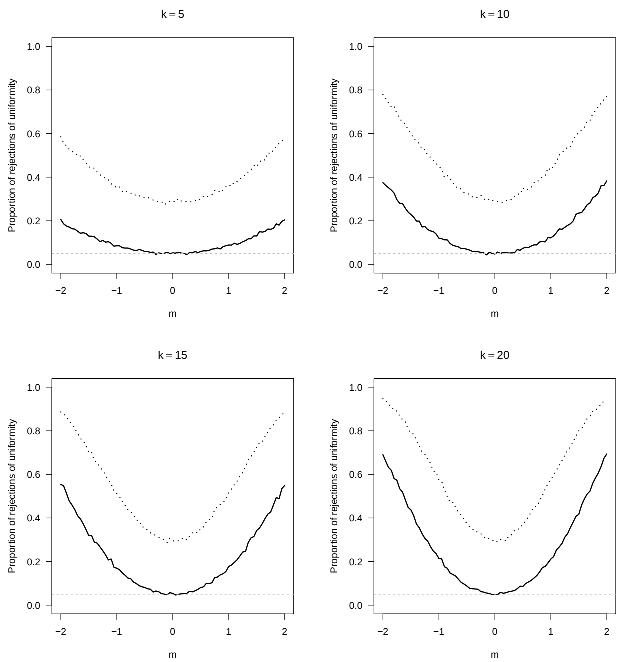

Fig. 1. Proportion of rejections of the null hypothesis of uniformity using Kolmogorov-Smirnov's test (dotted line corresponds to the augmented sample of size $2k$).

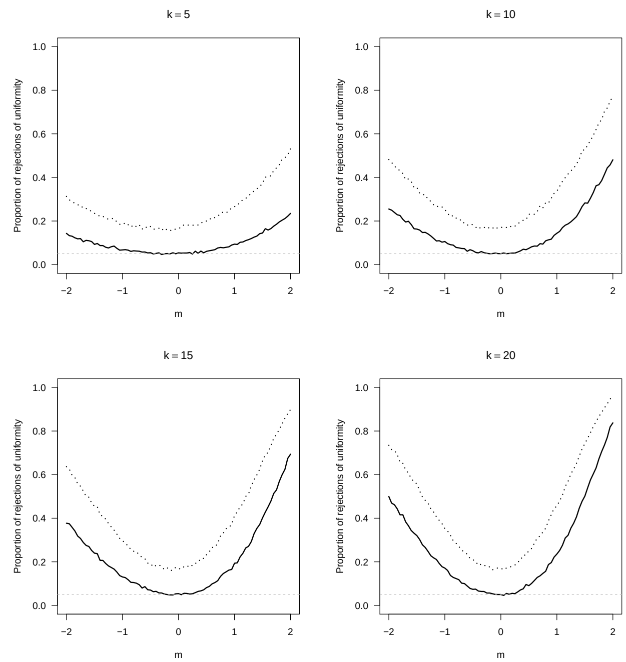

Fig. 2. Proportion of rejections of the overall null hypothesis using Fisher's test in a setting of combined p-values (dotted line corresponds to the augmented sample of size $2k$).

VI. Conclusion

The random variable V defined in (3) is very versatile, and useful namely in computational studies.

When $X \sim Mendel(m)$ in (3) the resulting V Mendel variable always has an heavier uniform component than X itself. This is drastically so when Y is also a Mendel variable, as shown in the Corollary of Section 3, and as a consequence using $Y \sim Mendel(p)$ in (8) to augment samples is pointless.

On the other hand, when the available sample of p-values is small and suspicious in the sense that some of them can be fake p-values, the use of the more general result in the Theorem of Section 3 to augment the sample size, requiring only that $\mathbb{E}[Y] \approx 1$, is sensible, since this can definitively increase power either in testing uniformity or the combined p-value.

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