Control of Invariants in Quasi-Polynomial Models

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***Abstract*—An approach to control of invariant sets of quasi-polynomial systems in the presence and absence of bounded disturbances or bounded uncertainty in the model is proposed. The control strategy is based on introduction of an invariant functional for uncontrolled system and posing the control task as achieving the desired value of the invariant functional by means of control. The design is based on the reduction to the generalized Lotka-Volterra system and employing the speed-gradient control method.**

***Index Terms*—Invariants, nonlinear control, stability**

I. Introduction

Quasi-polynomial systems represent an important type of mathematical models because a wide class of smooth nonlinear systems can be represented in a quasi-polynomial form [1], [2]. In turn, quasi-polynomial systems can be reduced to generalized Lotka-Volterra form [3], [4] that is a well known model for description of multispecies populations [9]. Besides, other standard modeling forms of biological or biochemical interest, such as S-systems or mass-action systems, are naturally embedded into the generalized Lotka-Volterra form [3]. Generalized Lotka-Volterra model has been proved useful in the analysis and control of the systems described by a set of differential and algebraic equations. However most of existing results are related to stabilization of equilibrium points [2], [5], [6],

In a number of interesting applications the problem of control of invariants arises [10], [11], [12], [13]. A feature of control of invariants is in that the goal limit set is a manifold rather than a point. Therefore some set stability problems may arise. For a class of multispecies Lotka-Volterra systems a solution for an invariant control problem based on speed-gradient (SG)-method was proposed in [16].

In this paper an approach of [16] is extended to a class of quasi-polynomial systems. We present a control strategy that can improve stability and robustness of quasi-polynomial systems in the presence and absence of bounded disturbances or bounded uncertainty in the model. In this way, we introduce an invariant functional and pose the control task as achieving a desired value of the invariant functional.

Problem formulation is given in Section 2. Section 3 describes the control design. Section 4 and 5 provide formulations and proofs of the closed loop system properties in the absence and presence of bounded disturbances or bounded uncertainty in the model respectively.

II. Problem Formulation

A. Mathematical Model

Quasi-polynomial model is described by the following system of differential equations

$$\dot{y}_j = y_j \left(L_j + \sum_{i=1}^{m} A_{ji} \prod_{k=1}^{n} y_k^{B_{ik}} \right),\ j = 1, \ldots, n, \quad (1)$$

where $y \in int\left(R_+^n\right)$, $A \in R^{n \times m}$, $B \in R^{m \times n}$, $L_i \in R$, $j = 1, \ldots, n$. Besides $L = (L_1, \ldots, L_n)^T$. It is assumed that $rankB = n$ and $m \geq n$.

In [7] the authors show that the model (1) can be reduced to the generalized Lotka-Volterra also known as classical model of multispecies populations [8], [9]:

$$\dot{x}_i = x_i \left(N_i + \sum_{j=1}^{m} M_{ij} x_j \right),\ i = 1, \ldots, m, \quad (2)$$

where

$$M = B \cdot A,\ N = B \cdot L, \quad (3)$$

and x_i is presented by

$$x_i = \prod_{k=1}^{n} y_k^{B_{ik}},\ i = 1, \ldots, m. \quad (4)$$

Let us choose initial values of variables x_i, $i = 1, \ldots, m$ according to initial values of variables y_j, $i = 1, \ldots, m$ and equations (4). Then dynamics of the multispecies populations (2) are equivalent to dynamics of the original quasi-polynomial model (1). Since the system (2) includes the variables y_i, $i = 1, \ldots, n$, the stability of this system implies stability of the original system (1).

Introduce control inputs $u_l,\ l = l_* + 1, \ldots, m,\ l_* \geq 1$ in (2). The controlled model of multispecies populations introduced in [16] is as follows:

$$\begin{cases} \dot{x}_i = x_i(t) \cdot \left(N_i + \sum_{j=1}^{m} M_{ij} x_j(t)\right),\ i = 1, 2, \ldots, l_* \\ \dot{x}_l = x_l(t) \cdot \left(N_l + \sum_{j=1}^{m} M_{lj} x_j(t) + u_l(t)\right),\ l = l_*, \ldots, m. \end{cases} \tag{5}$$

B. Invariant Functional

Assume that there exists at least one positive equilibrium in the uncontrolled system (2) for some values of the system parameters:

$$x_i = n_i > 0,\ i = 1, \ldots, m, \tag{6}$$

and the quantities $M_{ij}, i \neq j$ evaluating the type and intensity of the interaction between i-th and j-th variables form an antisymmetric matrix

$$M_{ii} = 0,\ M_{ij} = -M_{ji},\ i, j = 1, \ldots, m. \tag{7}$$

then the function

$$V_{qp}(x) = \sum_{i=1}^{m} n_i \left(\frac{x_i}{n_i} - \log \frac{x_i}{n_i}\right), \tag{8}$$

is an invariant of (5) for $u_l = 0,\ l = l_* + 1, \ldots, m,\ l_* \geq 1$ [8]. Besides, Hessian matrix of $V_{qp}(x)$ is positive definite and, therefore, $V_{qp}(x) > V_{qp}(n)$ for $x \neq n$. Hence $V_{qp}(x)$ can measure the amplitude of oscillations. Below it is used to achieve the desired amplitude of oscillations.

Introduce the control goal as an achievement of the desired level of the quantity $V_{qp}(x(t))$ as $t \to\propto$:

$$V_{qp} \to V_{qp}^*, t \to\propto . \tag{9}$$

If $V_{qp}^* = V_{qp}(n) = \min V_{qp}(x)$, then the goal (9) means achievement of the equilibrium $x = n$. In the case $V_{qp}(n) < V_{qp}^* < V_{qp}(x(0))$ achievement of the goal (9) means decrease of the oscillations level. If $V_{qp}^* > V_{qp}(x(0))$, then achievement of the goal (9) corresponds to the growth of the oscillations intensity. The problem is to find control function $u(t)$ in (5), ensuring achievement of the control goal (9).

III. Control Design

Apply the speed gradient (SG) method [14] to solve the problem. To this end introduce the so called goal function Q:

$$Q(x) = \frac{1}{2}\left(V_{qp}(x) - V_{qp}^*\right)^2. \tag{10}$$

In order to achieve the goal (9), it is necessary and sufficient that $Q(x(t)$ converges to zero as $t \to \infty$. According to the SG method one needs to evaluate A) derivative (speed of change) of Q with respect to the system (5) and B) the gradient of $\dot{Q}$ with respect to u.

Calculation of the time derivative of Q with respect to system (5) yields:

$$\dot{Q}(x, u) = \left(V_{qp}(x) - V_{qp}^*\right) \sum_{l=l_*}^{m} \left(x_l(t) - n_l\right) u_l. \tag{11}$$

Partial derivatives $\dot{Q}(\cdot)$ with respect to u_l are evaluated as follows:

$$\frac{\partial}{\partial u_l} \dot{Q}(x, u) = \left(V_{qp}(x) - V_{qp}^*\right)\left(x_l(t) - n_l\right),\ l = l_*, \ldots, m. \tag{12}$$

According to the SG method the control action is chosen as follows:

$$u_l(t) = -\gamma_l \left(V_{qp}(x) - V_{qp}^*\right)\left(x_l(t) - n_l\right), \tag{13}$$

where $\gamma_l > 0$, $l = l_*, \ldots, m$, $l_* \geq 1$.

IV. Control of quasi-polynomial systems in the absence of bounded disturbances or uncertainty

The first result of this section is the following statement.

Theorem 1. *Assume that there exists an equilibrium in the system (5) such that the conditions (6), (7) hold.*
Then either the algorithm (13) provides the goal (9), or the quantities of the controlled variables x_l tend to their equilibrium values n_l, $l = l_*, \ldots, m$, $l_* \geq 1$.
If the desired level $V_{qp}^* \geq V_{qp}^e$, where V_{qp}^e is a minimum of the invariant, and $V_{qp}(x(0)) > V_{qp}^e$, then the control goal (9) is achieved.

Proof.

Consider the time derivative of the goal function Q (11):

$$\dot{Q}(x, u) = -2\gamma Q \sum_{l=l_*}^{m} \left(x_l - n_l\right)^2 \leq 0. \tag{14}$$

Since Q does not increase, there exists a finite limit of $Q(t)$ as $t \to\propto$. Denote it as $\overline{Q}$. Suppose the goal (10) does not hold. Then $\overline{Q} > 0$. Hence $Q(t) \geq 0$ for all $t \geq 0$ and

$$\dot{Q}(x, u) = -2\gamma \overline{Q} \sum_{l=l_*}^{m} \left(x_l - n_l\right)^2 \leq 0. \tag{15}$$

Integration (15) yields

$$\begin{aligned} &0 \leq Q\left(x(t), u(t)\right) \leq Q\left(x(0), u(0)\right) - \\ &-2\gamma \overline{Q} \sum_{l=l_*}^{m} \int_0^t \left(x_l(s) - n_l\right)^2 ds \leq 0. \end{aligned} \tag{16}$$

Therefore

$$\sum_{l=l_*}^{m} \int_0^t \left(x_l(s) - n_l\right)^2 ds <\propto . \tag{17}$$

Since the integrand is nonnegative and uniformly continuous, it converges to zero according to Barbalat Lemma [15], that is

$$x_l(t) \to n_l,\ t \to \infty,\ l = l_*, \ldots, m. \tag{18}$$

Thus either the algorithm (13) provides the control goal (5), or a number of the controlled variables $x_l(t)$ converges to its equilibrium n_l, $l = l_*, \ldots, m$, $l_* \geq 1$.

The above assertion implies that the function $V_{qp}(x)$ either achieves the desired level V_{qp}^*, or converges to $V_{qp}(n) = V_{qp}^e$. Therefore at $x_i = n_i$, $i = 1, \ldots, m$ the function $Q(x) = 0.5\left(V_{qp}(x) - V_{qp}^*\right)^2$ has its minimum. Thus for all $t \geq 0$ $V_{qp}(0) \geq V_{qp}^e$. Provided that $V_{qp}(0) = V_{qp}^e$, the system is always in its equilibrium, i.e. to achieve the control goal for $V_{qp}^* \geq V_{qp}^e$ it is necessary $V_{qp}(0) > V_{qp}^e$ ◄

Remark. In Theorem 1 it is supposed that the system (2) has at least one positive equilibrium for some values of its parameters. For a nonsingular matrix composed of M_{ij} we always can choose values of the birth rate N_i such that (6) holds [17]. For a nonsingular matrix composed of M_{ij} positivity conditions depending only on M_{ij} were found in [8].

V. Control of quasi-polynomial systems in the presence of bounded disturbances or uncertainty

A. Control of nonlinear systems in the presence of bounded disturbances or uncertainty

Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)u + \eta, \\ y = h(x) \end{cases} \tag{19}$$

where $x \in X \subset R^n$ is a vector of state variables, $u \in U \subset R^m$ is a vector of control actions, $y \in R^p$ is an output vector. The vector $\eta \in R^n$ characterizes disturbances or uncertainty of the system (19). X, U are open sets in the space of dimension n and m accordingly; g is a $n \times m$ matrix function; f, h are smooth vector functions of dimension n and p accordingly. Moreover, $h(x)$ is an invariant function of (19) by $u = 0$.

Assume that in the system there exists an unique solution $x(t)$ for all $x(0) \in X$ and $u \in U$, and this solution is defined on $[0, +\infty)$ and entirely contained in the set X.

Introduce a control goal as achieving such quantity of the invariant $h(x)$ that will be the closest one to the desired value with the required accuracy:

$$\overline{\lim_{t\to\infty}}\, Q\left(x(t)\right) \leq C_Q, \tag{20}$$

where $Q = y^2$.

Apply the speed gradient (SG) method [14] to solve this problem. As a goal function take the function Q:

$$u = -\tilde{\gamma} \nabla_u \dot{Q} = -\gamma y^T \nabla h^T. \tag{21}$$

The second result of this paper is the following statement.

Theorem 2. *Suppose that the following conditions on the system (19) hold:*

- $f, g, h \in C^1$.
- $\|\eta(t)\| \leq C_\eta$.
- $L_f h(x) = 0$, *i.e.* $h(x)$ *is an invariant function in* (19) *by* $u = 0$.
- *There exists* $\xi > 0$ *such that a set* $Q_\xi = \{x \in R^n : Q(x) \leq \xi\}$ *is compact.*
- $x(0) \in Q_\xi$.
- $\forall x \in Q_\xi$ $\|h(x)^T \nabla h(x)^T\| \leq C$.
- *The minimum eigenvalue of the matrix* $A(x)^T A(x)$ *is uniformly positive, where* $A(x) = \nabla h(x)^T g(x)$*:*

$$\epsilon = \inf_{X \in R^n} \lambda_{min}\left(A(x)^T A(x)\right) > 0.$$

Then the designed control algorithm (21) *will provide the control goal* (20) *with* $C_Q = 2CC_\eta/\epsilon$*.*

Proof.

Consider the time derivative of the goal function Q along trajectories of (19):

$$\begin{aligned} \dot{Q} = \tfrac{\partial Q}{\partial x}\dot{x} = 2y^T \nabla h^T (f + gu + \eta) = \\ 2y^T \nabla h^T f + 2y^T \nabla h^T gu + 2y^T \nabla h^T \eta. \end{aligned} \tag{22}$$

According to the first condition of Theorem 2 the first term of (22) is 0. Denote the second and third items of (22) as R_1, R_2 and estimate them:

$$R_1 = 2y^T \nabla h^T gu = y^T \left[\left(\nabla h^T g\right)\left(\nabla g^T h\right)\right] y = y^T A^T A y, \tag{23}$$

where $A = \nabla h^T g$. According to the fifth condition of Theorem 2 we obtain

$$A^T A \geq \epsilon I, \tag{24}$$

and, therefore

$$R_1 \leq -\epsilon \|y\|^2 = -\epsilon Q. \tag{25}$$

According to the first and forth conditions of Theorem 2 the functions f, g, h are bounded in the compact set Q_ξ, and ξ is bounded according to the second condition of Theorem 2. Therefore

$$R_2 = 2y^T \nabla h^T \eta \leq CC_\eta, \tag{26}$$

where C is a positive constant such that $\left\|2y^T \nabla h^T\right\| \leq C$. Then time derivative of Q

$$\dot{Q} \leq -\epsilon Q + CC_\eta. \tag{27}$$

that implies

$$\overline{\lim_{t\to\infty}}\, Q(x) \leq \frac{CC_\eta}{\epsilon}. \tag{28}$$

Thus, if the system (19) has bounded disturbances or uncertainty, the controls (21) limit the function Q, although the controls do not result in tending function Q to zero, and the upper estimate for the function Q is (28) ◄

B. Quasi-Polynomial Model in the presence of bounded disturbances or uncertainty

Quasi-polynomial model with bounded disturbances or uncertainty is presented by the system

$$\dot{x}_i = x_i(t)\cdot\left(N_i + \sum_{j=1}^{m} M_{ij}x_j(t) + u_i(t)\right) + \eta_i,\ i = 1,\ldots,N, \quad (29)$$

where $\eta = (\eta_1,\ldots,\eta_N)^T$ is a vector containing disturbances or uncertainty.

Introduce a control goal as achieving such quantity of the invariant (8) that will be the closest one to its desired value V^*_{qp} with the required accuracy:

$$\overline{\lim_{t\to\infty}}\, h^2\left(x(t)\right) \le C_{V_{qp}}. \quad (30)$$

where $h(x) = V_{qp}\left(x\right) - V^*_{qp}$.

Apply the control algorithm (13) based on the speed gradient method to achieve the control goal (30).

The following result holds.

Theorem 3. *Suppose in the system* (29) *the conditions hold:*

- $\|\eta(t)\| \le C_\eta$.
- *There exists* $0 < \xi < \left(V^*_{qp}\right)^2$ *such that a set* $Q_\xi = \{x \in R^n : Q(x) \le \xi\}$ *is compact.*
- $x(0) \in Q_\xi$.
- $\forall x \in Q_\xi\ \|h(x)^T \nabla h(x)^T\| \le C$.
- *Take*

$$\epsilon = \inf_{i;\ x\in Q_\xi} x_i^2 \cdot \sum_{i=1}^{N}\left(1 - \frac{n_i}{x_i}\right)^2.$$

Then the algorithm (13) *will provide the goal* (30) *with* $C_{V_{qp}} = 2CC_\eta/\epsilon$.

Proof.

Theorem 3 is a consequence of Theorem 2. Indeed, in Theorem 3 all requirements from Theorem 2 except the sixth one hold. We have to verify this requirement, namely that the minimum eigenvalue of the matrix A^TA is uniformly positive, where the matrix $A = \bigtriangledown hg^T$.

For the system (29) functions $h(x),\ g(x)$ are as follows

$$h(x) = \sum_{i=1}^{N}\left(\frac{x_i}{n_i} - \log\frac{x_i}{n_i}\right) - W^*. \quad (31)$$

$$g(x) = \left(x_1,\ldots,x_N\right)^T. \quad (32)$$

Then

$$\bigtriangledown h(x) = \left(\tilde{h_1},\ldots,\tilde{h_N}\right)^T,\ \tilde{h_i} = 1 - \frac{x_i}{n_i},\ i = 1\ldots,N. \quad (33)$$

$$A(x)^TA(x) = g(x) \bigtriangledown h(x)^T \bigtriangledown h(x)g(x)^T = \sum_{i=1}^{N}\left(1 - \frac{x_i}{n_i}\right)^2 \cdot diag\{x_i^2\}_{i=1}^N. \quad (34)$$

Therefore the eigenvalues of the matrix $A(x)^TA(x)$ are

$$\lambda_i = x_i^2 \cdot \sum_{i=1}^{N}\left(1 - \frac{x_i}{n_i}\right)^2. \quad (35)$$

Thus, all eigenvalues of the matrix $A(x)^TA(x)$ are strictly positive. Therefore the system (29) satisfies all requirements of Theorem 2 ◄

VI. Conclusion

An approach to control of invariant sets of quasi-polynomial systems in the presence and absence of bounded disturbances is proposed. The control strategy is based on introduction of an invariant functional and posing the control task as achieving a desired value of the invariant functional. The design is based on the reduction to the generalized Lotka-Volterra system control. The proposed method may improve stability of the closed loop system and its robustness under action of bounded disturbances or under bounded uncertainty in the model. To implement the proposed algorithm an instant information exchange between different agents (species) is needed. In some cases it may be implemented based on Distributed Ledger Technology.

Further research may be devoted to application of the proposed algorithms to control of various biological or biochemical systems and numerical examination of the designed systems behavior. Examples of such system models can be fount, e.g. in [1].

Another avenue of research is study of speed-gradient algorithms for modeling of the biological evolution based on maximum entropy principle and its dynamical speed-gradient version [18].

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