Road map for Bifurcations of Real Rational maps

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***Abstract*—With the help of a proper parameter space $P_{a,b}$, defined for the class of real rational maps (1), in this work, we define lines in the form $b = \varphi(a)$, that will be used as roads in a traffic map, which will contribute to a better understanding of their behaviour, under iteration. This family of maps have a very interesting dynamic, where we can confirm the existence of several bifurcation types. Using tools, from Combinatorial Dynamics, Entropy and Bifurcation Analysis, with common use in Low Dimension Dynamical Systems studies, it is shown that these roads clearly depend on the relationship between variables a and b, highlighting some important aspects of this relationship, which help to describe the dynamics of map (1).**

$$f_{a,b}(x) = 1 + \frac{b-a}{x^2-b}, \quad b < a, \quad b < 1 \tag{1}$$

***Index Terms*—Real Rational Maps, Iteration, Bifurcation**

I. Introduction

Discrete time dynamical systems generated by iterated maps appear in many scientific areas, such as economics, engineering, and ecology. To understand better the behaviour of these systems is used, frequently, some results derived from bifurcation analysis, establishing some order in chaotic events, classifying possible behaviours, whose may explain computational simulation results, with different values of control parameters.

The notion of iterated function system was introduced by M. F. Barnsley and S. Demko, in 1985, but the concept is usually attributed to Joan P. Hutchinson. According Edward R. Vrscay the idea is traced further back to the works of Leggett and Williams, who studied fixed points of contractive maps finite composition. Iterated function systems are interacting with many fields of mathematics. For example, they are useful for creating fractals, learning models, interesting probability distributions and analysing stochastic processes with Markovian properties.

In this paper it will be presented some numerical and geometrical results, supported by high and extensive analytical calculus, but not fully shown in this paper, due to size and complexity usually found in real rational maps, under iteration.

Let $f^n_{a,b}(x)$ be the $n-iterate$ of $f_{a,b}$, *i. e.*, the map composition, by itself, $n-times$. The sequence

$$\{x_i\}_{i=0,1,\ldots,n} = \{x_0, x_1, ..., x_n\}$$

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is the orbit of x_0, under iteration by $f_{a,b}$. It means that $x_{i+1} = f_{a,b}(x_i)$, $i = 0, 1, \ldots, n$. Each solution $x = \xi$ of $f^n_{a,b}(x) = x$, using fixed parameters $a = a_0$ and $b = b_0$, is designated fixed point of order n for f_{a_0,b_0}. These values, under iteration by f_{a_0,b_0}, are invariant. They can be classified as attractors if $|f'_{a,b}(\xi)| < 1$, repulsors if $|f'_{a,b}(\xi)| > 1$ and neutral if $|f'_{a,b}(\xi)| = 1$. The solution set of $f'_{a,b}(x) = 0$ is the critical set of $f_{a,b}$, where we will include the values $x = \pm\infty$. In this family of maps (1), by a simple graphic observation, we can see that $\underset{x\to\pm\infty}{lim} f_{a,b}(x) = 1$. So, under iteration of $f_{a,b}$, the values present in some neighbourhood of infinite, have the same behaviour of the value $x = 1$, under iteration. It is now obvious that the singularities of $f_{a,b}$, $x = \pm\sqrt{b}$, under iteration, will have also the same behaviour of the value $x = 1$, since $f_{a,b}(\pm\sqrt{b}) = \infty \Leftrightarrow f_{a,b}(f_{a,b}(\pm\sqrt{b})) = f_{a,b}(\infty) = 1$, so we will use the orbit $x = 1$ to represent the orbit of $x = \pm\infty$ and $x = \pm\sqrt{b}$. If, by any chance, the orbit of $x = 1$ would be periodic then we say that the orbits of $\pm\infty$ and $\pm\sqrt{b}$ are eventually periodic.

In classical low-dimension dynamics, as the study of *m*-modal maps under iteration, classified as interval maps [1] and [4], the analysis of critical orbit set is enough to have a full description of the map dynamics [4]. And the most important orbits, in continuous maps, are the ones with period 3, due to its connection to Sharkovsky's theorem, as shown very deeply in chapter 2 of [1].

Since our map (1) is discontinuous, and real, in the last decades small attempts where made to develop some consistent theory similar to the one developed to continuous interval maps in [4], but so far with no any relevant progress. We have excellent contributions from James Yorke [5] and Laura Gardini [6], among others referenced by these authors, attempts to minimize the damage caused by the presence of singular values, but the full description of the real rational maps dynamics is a stronghold very hard to conquer, even with the use of emerging computational tools of 21st century allied to the newest analytic tools. But one idea is clear, if we cannot deal very well with the singularities, at least we can use the continuous part of the function and make some restrictions to the dynamical domain and compare the findings, building a Scottish quilt of knowledge that can be close to that should be the full dynamical description of the real rational map.

Since $f'_{a,b}(x) = 0 \Leftrightarrow x = 0$ and $\underset{x\to\pm\infty}{lim} f'_{a,b}(x) = 0$ then the critical set of $f_{a,b}$ will be $\Lambda = \{0, \infty\}$. Assuming that the critical orbits are the ones produced by the critical values, and following the road of discovery like Milnor and Thurston did in [4], for continuous maps, then we will try to reveal the dynamical secrets of this class of maps $f_{a,b}$. To do that we used some computational work, and construct the proper analytical tools to prove some results. Numerically, we create a process to identify regions in $P_{a,b}$, defined as the parameter space for $f_{a,b}$, where, for some fixed $a = a_0$ and $b = b_0$ we can find periodic orbits for $x = 0$ and $x = 1$. To do an organized search we will study the map's behaviour following the lines $b = \varphi(a)$, the paths or roads, with $a \in I$ (see section III). We define the set $\Sigma = \{(a, b) \in I : f^n_{a,b}(0) = 0 \vee f^n_{a,b}(1) = 1, n \in \mathbb{N}\}$, where to any fixed pair $(a, b) = (a_0, b_0)$, we can find roads in the parameter set $P_{a,b}$, such the map f_{a_0,b_0} will have periodic super-stable orbits, under iteration. Since we will work, mostly, with the geometric view of the orbits, it is usual to call them trajectories.

Studying the geometry of $P_{a,b}$ and Σ, it is our goal to show that $f_{a,b}$, as a piecewise differentiable map, presents some behaviour similar to the one exhibited by bi-modal and one-modal class of maps studied by Milnor and Thurston [4], among so many other authors, that followed their work. To fulfil the goal, we use techniques derived from combinatorial dynamics, such as Bifurcation Analysis, Entropy Study and Interpretation of Lyapunov Exponents value [3], to study the relation between periodic orbits and the behaviour of map (1) under iteration.

II. Lyapunov and Bifurcation Theory

Chaotic behaviours are characterized by a high sensitivity to initial conditions: Starting from arbitrarily close to each other, the trajectories rapidly diverge.

The map (1) is discontinuous, then the results, already known for continuous maps, cannot be applied to this map's family, but we can use some of them as a start point to understand its dynamics. One of these tools are the Lyapunov Exponents, integrated in a very large field of research known as Lyapunov Theory. The connection between this Theory and the study of the dynamics of real maps is, undoubtedly, very important, since help to understand the connection between analytic results and computational. The power of Lyapunov Theory comes from the fact that it is used to make conclusions about the dynamics of a system, without finding exactly the values of the trajectories, saving computational time and endless analytic efforts. Young [7] and Katok [3] have a splendid description of use and properties of Lyapunov exponents.

For a function $f(x)$, each trajectory $\{x_i\}$ have the Lyapunov Exponent defined as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} ln \left| f^{'}(x_i) \right| \qquad (2)$$

Since λ is the same for all x_i on the basin of attraction of ξ, if ξ is an attractor, the sign of λ defines the attractor type. If $\lambda < 0$ we are in the presence of limit cycle or stable fixed points; If $\lambda > 0$ we have chaotic attractors. For bifurcation values of the function, we will have $\lambda = 0$, and $\lambda \to \infty$ for values where $f(x)$ have super stable orbits. Lyapunov Exponent are also used to calculate an estimative to the Topological Entropy, from which we can obtained detailed information about the orbit behaviour. See [7] for a more complete description.

The bifurcation of a function is characterized as being a splitting of a specific orbit, occurring with the modification of a parameter that controls the function. For example, for $f_\lambda(x) = \lambda x(1 - x)$, with the change of parameter λ, we will assist to a double period bifurcation, with periodic orbit $n = 2$ splitting to $n = 4$, then goes $n = 8$, and so on. But it can occur also the splitting from $n = 1$ to $n = 3$, then $n = 7$, and so on, like the maps studied by Laura Gardini in [6].

The map (1) have parameters a and b, and for certain values of the pair (a, b), the structure of fixed points and periodic orbits changes. In the same way as the maps with only parameter, we define this change as a bifurcation. The graphic, where we can analyse, geometrically, the period variations regarding the parameter change is called Bifurcation Diagram. To build the bifurcation diagram of (1) we need to make $b = \varphi(a)$, in order to transform $f_{a,b}$ in a function of one parameter only. There are many types of bifurcations present in a simple bifurcation diagram for $f_{a,\varphi(a)}$, and we will explore it in section (IV), as we can see, for example, in figure 4. We can find *saddle-node bifurcations*, occurring when a pair of fixed points appears in a region where there were none, with one stable fixed point and one unstable fixed point; *period-doubling bifurcation*, characterized by the loss of stability of the original fixed point, the period doubles, and the nature of attractor changes; *border-collision bifurcations*, as described in detail by Helena E. Nusse and James Yorke in [5] and complemented by Roya Makrooni, Farhad Khellat and Laura Gardini in [6] is mainly characterized by a suddenly change of one fixed point attractor in a *m-piece* chaotic attractor. Also, we can find the reverse bifurcation phenomena.

III. Parameter space $P_{a,b}$ for $f_{a,b}(0)$.

To study the behaviour, under iteration, of the map (1) we need some simple results about the variables domain, in order to build a parameter space where we will get useful information. In [2], we can found complementary data about the map (1).

We establish the domain for the parameters a and b as the set

$$I = \left\{ (a, b) \in \mathbb{R}^2 : 1 - \frac{2\sqrt{3}}{9} < a < 1 + \frac{2\sqrt{3}}{9}, b < a, b < 1 \right\}.$$

As we can check in [4], due to the Sharkovskii theorem, the orbits of period $n = 3$, of the critical points, assumes in the dynamics of a continuous map a very important role, since their existence in continuous maps assures the existence of all others orbits. So, will use, as reference, the period 3 orbit of the critical values $x = 0$ and $x = 1$.

As explained before, whenever a value, under iteration, falls in a neighbourhood of some $f_{a,b}$ discontinuity, the forward image will be ∞, and the next iteration will be trapped in the orbit of $x=1$. For our map (1), the lines $b=\varphi(a)$, where this phenomena occurs, will play an important role in the function dynamics, since the computational calculus will tend to be unstable near these lines. Solving the equation $f^3_{a,b}(1)=1$, we will have two possible lines: $b=a$, that reduces the map to a trivial one, and

$$b=\frac{1}{3}\left(2-\left(\frac{2}{\omega}\right)^{1/3}-\left(\frac{\omega}{2}\right)^{1/3}\right),$$

with $\omega=-25+54a-27a^2+\sqrt{-4+(-25+54a-27a^2)^2}$. For $4+\left(25-54a+27a^2\right)^2=0$, will have $a=\frac{1}{9}\left(9\pm 2\sqrt{3}\right)$. These values are the ones used to set the range for a in I.
We define the parameter space

$$P_{a,b}=\{(a,b)\in I: f^n_{a,b}(x)=x, n=3,4,\ldots\},$$

represented in figure 1.

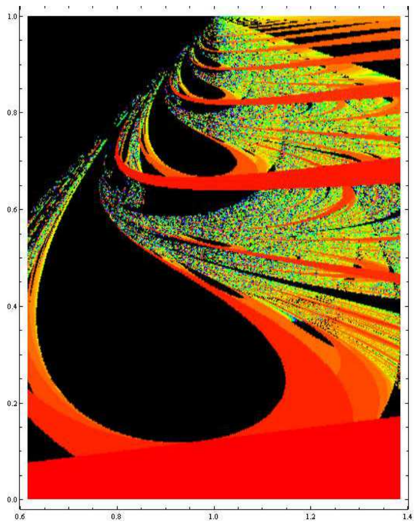

Fig. 1. Parameter space $P_{a,b}$ with $n<120$ for $f^n_{a,b}(x)$.

It appears to have fractal properties, since we can see a process of self-similarity. Each one of the big black regions, after excluding the upper-left black region where $b>a$, are sets, designated by *n*-Bulbs in [2], geometric neighbourhoods of all solutions lines $b=\varphi(a)$ of the equation $f^n_{a,b}(0)=0$, which each pair (a,b) produces maps with critical super-stable orbits with period n.
We can, in $P_{a,b}$, identify important lines, see figure 2, where the solution line $b=\varphi(a)$ of $f^3_{a,b}(0)=0$ is coloured in white; the solution of $f^3_{a,b}(0)=-\sqrt{b}$ in yellow; the solution of $f^3_{a,b}(0)=\sqrt{b}$ in green and the solution of $f^3_{a,b}(1)=1$ in blue.

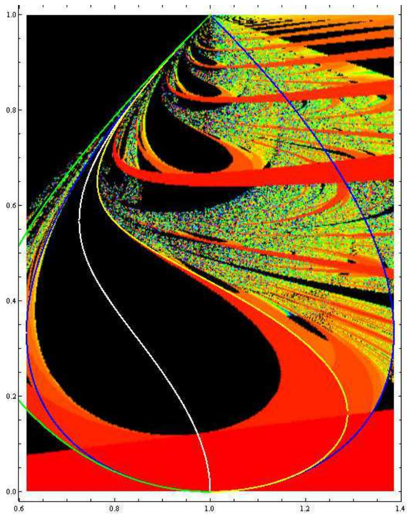

Fig. 2. Relation between $P_{a,b}$ and the lines $b=\varphi(a)$ in $f^3_{a,b}(p_1)=p_2$, with $p_1,p_2\in\{0,\ 1,\ \pm\sqrt{b}\}$

Definition 1. *Let the solution line $b=\varphi(a)$ of the equation $f^3_{a,b}(1)=1$, such that all the points are included in I. This line is the border of a region that we will define as the locus L_f.*

L_f will help us to understand the diagrams in the next section.

IV. Bifurcations Exploration

Now, we will transform our map (1) in one parameter map.

Let $b=\varphi(a)$, with $(a,b)=(a,\varphi(a))\in P_{a,b}$ with $\varphi\in C^1$, then we will have

$$f_{a,\varphi(a)}(x)=f_a(x)=1+\frac{\varphi(a)-a}{x^2-\varphi(a)}=\frac{x^2-a}{x^2-\varphi(a)} \quad (3)$$

With this transformation we can start to explore the dynamics of (1) in the interior of L_f, analysing the bifurcation diagrams of the critical orbit $x=0$. We choose in this paper to explore just the cases where φ is a straight line with positive slope.
We can see in figure 3 the line $b=\varphi(a)=-1.14723+2.09677a$, in cyan, crossing all the basins of attraction of the super stable lines, and analysing the correspondent f_a bifurcation diagram, figure 4, we can identify at least one value $a=0.76$, where the orbit of the critical point $x=0$ will produce a periodic $n=3$ super stable orbit.

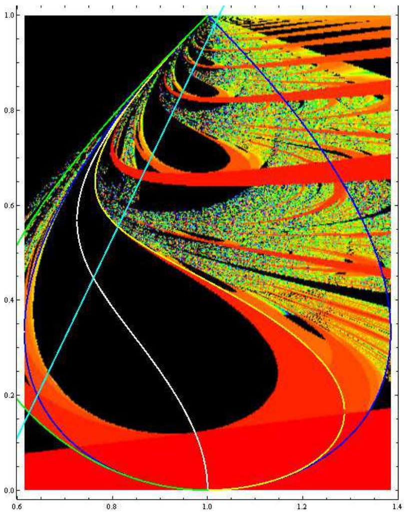

Fig. 3. Example line $b = -1.14723 + 2.09677a$

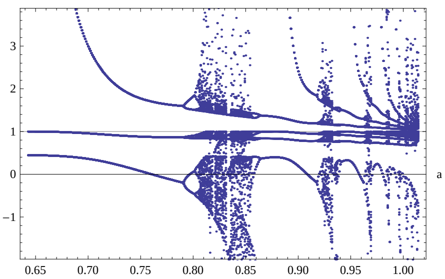

Fig. 4. f_a bifurcation diagram, $\varphi(a) = -1.147 + 2.096a$, $0.57 < a < 1.43$

Also we can observe intervals of stability for f_a and others where chaos prevail. For certain intervals of the value a we can identify phenomena like reverse bifurcations, double period bifurcations and, hidden among chaos windows, saddle-node bifurcations. Using Lyapunov Exponent diagram, figure 5 we can calculate the maximum value of topological entropy for $0.80 < a < 0.85$, that is, approximately $h = 0.409038$.

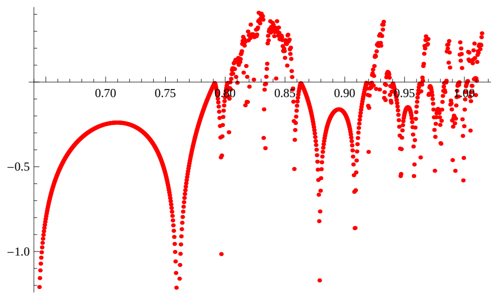

Fig. 5. f_a Lyapunov Exponents, $\varphi(a) = -1.147 + 2.096a$

The analysis of the bifurcation diagram is not sufficient to produce a deep study about the dynamics of function (3). Hidden, in intervals of supposed chaos, we can observe some regularity, like it happens in classic bifurcation diagrams for continuous maps. Also, to explore analytically the dynamics of f_a, it can be a very hard process due to the nature of rational maps iteration. Nowadays, most of the results, arising from the low dimension dynamics study, for real rational maps, are initial triggered by computational numeric calculus in association with a very deep knowledge of Implicit Theorem application. To avoid the analytic difficulties created by the conditions necessary to apply the Implicit Theorem, we show that the combined use of Lyapunov Exponent and Bifurcation diagrams, can be a great tool, providing a good initial approximation, on the search for intervals of chaos or regularity of f_a.
As an example, let $\varphi(a) = 0.02025 + 0.26625a$, the straight line in figure 6, and the bifurcation diagram in figure 7, that shows the behaviour of the orbit of $x = 0$, under iteration of f_a, with $0.57 < a < 1.43$.

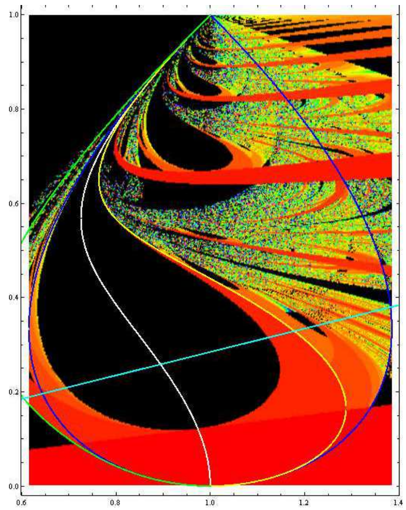

Fig. 6. $\varphi(a) = 0.02025 + 0.26625a$ crossing $P_{a,b}$

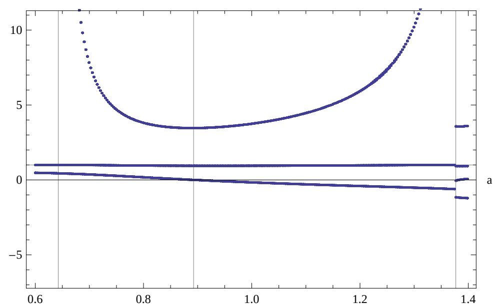

Fig. 7. Bifurcation diagram for f_a with $\varphi(a) = 0.02025 + 0.26625a$,and $0.57 < a < 1.43$

Close to the value $a = 0.8925$ we have a super stable orbit of the critical point and at $a = 1.377$ we have a border collision bifurcation. But what happens at $a = 0.6425$? Is it another border collision bifurcation? If yes, it is not so visible. Merging both diagrams, bifurcation an Lyapunov Exponent, we have the figure 8. As mentioned in section II, if $\lambda \to \infty$ then we have the presence of super stable orbit, and that is the case of $a = 0.6425$, and $a = 1.377$, where the orbit of the critical point $x = 0$ falls in an super stable orbit of the other critical point $x = \infty$. We can observe that these two values corresponds to the intersection of $\varphi(a)$ with the solution line of $f_a^3(1) = 1$. The only super stable orbit of $x = 0$ is at $a = 0.8925$, easily identified because it corresponds to the intersection of $\varphi(a)$, with the solution line of $f_a^3(0) = 0$, see figure 6. Also, in figure 8 we can see, for $1.2 < a < 1.3$, that we will have at least one value where $\lambda = 0$, revealing a region where we will find another bifurcation point, and it must be a period doubling or saddle-node bifurcation. So the points $a = 0.6425$ and $a = 1.377$ are border collision bifurcation points of $x = 0$, where the orbit undergoes in super stable orbit of $x = \infty$.

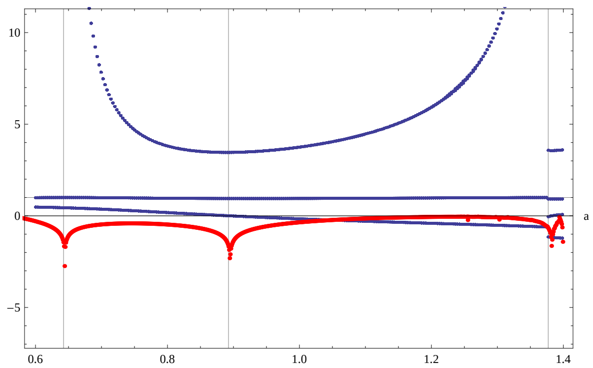

Fig. 8. Lyapunov Exponents (in red) and bifurcation diagram (in blue) for f_a with $\varphi(a) = 0.02025 + 0.26625a$, $0.57 < a < 1.43$, with vertical lines in the position where (2) assumes infinite values

Graphically, a border collision can be wrongly identified as a saddle-node bifurcation, but with the help of Lyapunov Exponents we can avoid this graphic confusion. Let's take another example, zooming $P_{a,b}$ to the region represented in figure 9.

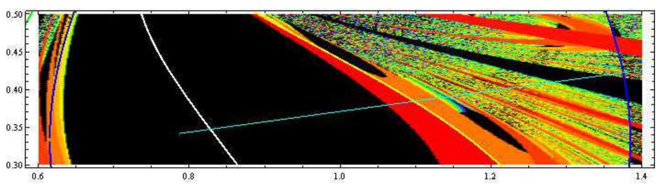

Fig. 9. Line $\varphi(a) = 0.23353 + 0.13721a$ crossing $P_{a,b}$

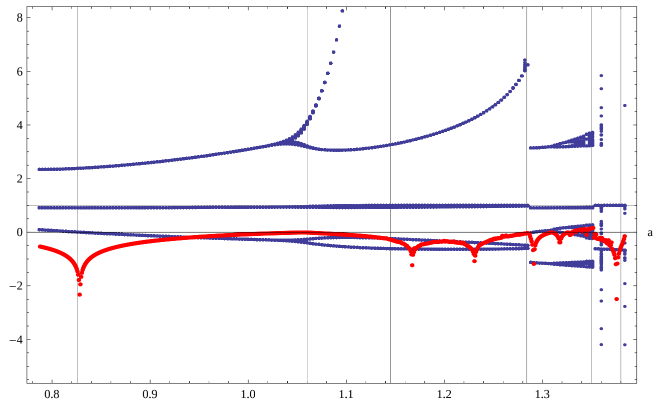

Fig. 10. Lyapunov Exponents (in red) and bifurcation diagram (in blue) for f_a with $b = 0.23353 + 0.13721a$, $0.786 < a < 1.3844$

Selecting $\varphi(a) = 0.23353 + 0.13721a$ we get a interesting f_a dynamic, as presented in figure 10. We have the presence of a super stable orbit, at $a = 0.826$, with $\lambda = \infty$, a double period bifurcation, at $a = 0.6425$, with $\lambda = 0$, and most important a bifurcation at $a = 1.284$, which at first glance, probably could be identified as a border collision bifurcation, but since $\lambda = 0$, at that position, then it must be a saddle node bifurcation, as also happens at $a = 1.35$, occurring the border collision bifurcation at $a = 1.38$.

V. Results

Using $P_{a,b}$ as a guide map, we can find the roads $b = \varphi(a)$, construct the bifurcation diagram, the Lyapunov Exponent diagram and write conclusions about the dynamic of map (1), related with the parameter a change. Clearly, the association between these two graphics is a powerful tool, allowing the researcher collect precious information, and since the hunt for some properties and new kind of bifurcations can be done, using numeric computational calculus, they can be the trigger for new ideas and a good start to initiate the analytic proof of the graphically observed phenomena.

As we show, in this work, only using the fundamentals of discrete dynamical systems, we can discover very easily, some special regions in $P_{a,b}$, where we can find, for f_a, well known behaviours, observed on the dynamics of continuous logistic maps and also in piecewise continuous m-modal maps, but also other behaviour not so common. We also remember that we focused our attention in L_f, and due to its fractal nature and self similarity it is easy to see that all the phenomena described graphically in last section for period 3 orbits also happens for all other periodic orbits.

Let's examine figure 11.

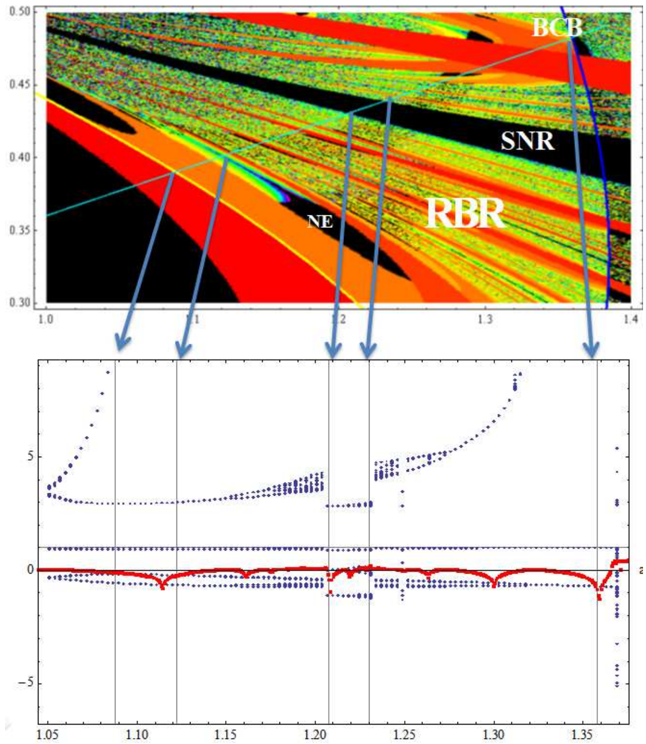

Fig. 11. Comparison between $P_{a,b}$ and the bifurcation and Lyapunov Exponents diagrams for the line $b = \varphi(a) = 0.0.01972 + 0.3404a$, with the presence of a reverse bifurcation with saddle node bifurcations at its centre.

We use $\varphi(a) = 0.01972 + 0.3404a$ and signalize the values $a \in \{1.088, 1.122, 1.2074, 1.2302, 1.3582\}$ with arrows. In 11 we enhance the region designated by **SNR**, in which borders the values a produce a saddle-node bifurcation. Indeed this kind of bifurcation will happen for $a = 1.2074$ and $a = 1.2302$. Also, we enhance the presence of a region designated by **RBR** where the values a on its south border will be the responsible for the appearance of a double period bifurcation, which one that goes in a reversion process inside of this region building a reverse bifurcation, until the north border when phenomena ends, entering region **SNR**. The BCB point, where the value a produces a Border Collision Bifurcation, is already signalized before and it is part of the solution line $f_a^3(1) = 1$. Another special region, designated by **NE**, is a region where the values at its south border starts a double period bifurcation, but at the north border, all the process reverts in a single point to a period order before doubling and then starts a reverse bifurcation process already inside **RBR**.

If we shift the line $\varphi(a)$, just enough to cross all 4 regions, we obtain an amazing representation of the dynamics of f_a, as represented in figure 12, and it is easy to identify the points where occur reverse bifurcation, saddle-node bifurcations and also border collision bifurcations.

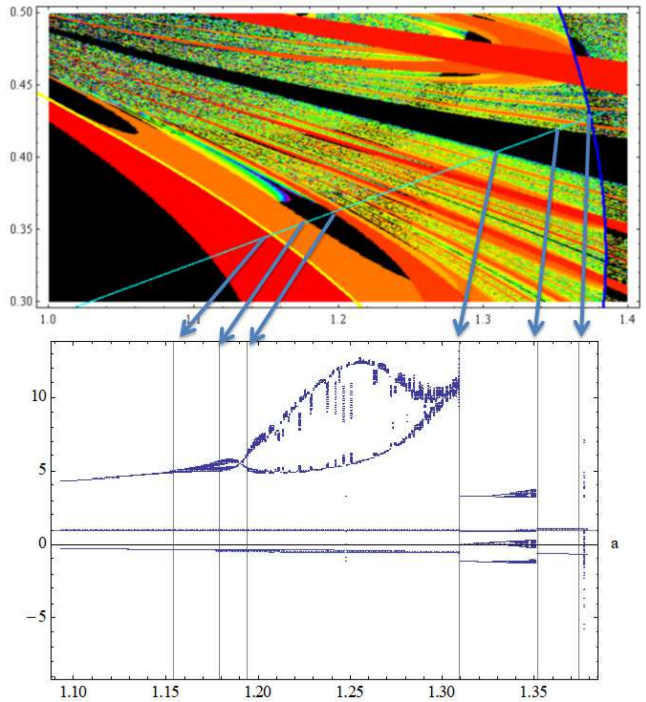

Fig. 12. Effects when the parameter a crosses the borders of regions NE, RBR and SNR, with a clear presence of a reverse bifurcation.

With the use of parameter space $P_{a,b}$ as a map to study the behaviour of family f_a, we encounter a fertile ground where can lead to discoveries related with the amazing properties of this family of maps, building proper roads. Further, with the adaptation and extension of some tools of piecewise continuous maps, the main goal is to prove that maps like $f_{a,b}$ exhibits a behaviour that resemble the one presented by the *m-modal* families.

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