A Remark on An Annuity Payment Structure

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Abstract **— We discuss a possible occurrence of annuity payments which follow a basic-hypergeometric progression, relative to the associated spot rate. We give explicit formulas for calculating the outstanding balance in a particular model where a single interest rate is replaced with a particular interest rates structure.**

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I. INTRODUCTION AND MAIN RESULT

An annuity is a series of payments, typically following a pattern, e.g., level (i.e., same every period). Annuities are used for calculation of loan amortization schedules, bond prices, and in insurance applications. If a loan in the initial amount L is being paid off by a series of level payments P over n years with annual effective interest rate of i, with payments made at the end of each year, then the outstanding balance of a loan at time k (expressed in years), where k is an integer, just after the k-th payment was made at the end of the $(k-1)$-st year, is

$$OB_k = Pa_{\overline{n-k}|i} = P \cdot \frac{1-(1+i)^{-(n-k)}}{i}, \qquad (1)$$

alternatively written as

$$OB_k = L(1+i)^k - Ps_{\overline{k}|i} = L(1+i)^k - P \cdot \frac{(1+i)^k - 1}{i}. \qquad (2)$$

(1) is called the *prospective method* of calculation of a loan balance, and (2) is termed the *retrospective method* of calculation (see [1]). In the above k and n are natural numbers.

Loan repayments are not always level in practical scenarios. One possible alternative is to pay l times the interest due the loan whose initial balance was L, where l is a parameter greater than 1. With an annual effective interest rate of i, the loan balance would be: L at time 0,

$$L - L(l-1)i = L(1-(li-i))$$

at time 1, then

$$L(1-(li-i))\,(1-(li-i)) = L(1-(li-i))^2$$

at time 2, etc., so that in each payment the balance of the loan is multiplied by the expression $(1-(li-i))$ and with balance at time k equal to

$$L\left(1-(li-i)\right)^k = L\left(1-(l-1)i\right)^k = = L\cdot\sum_{j=0}^{k}\binom{k}{j}\left(-(l-1)i\right)^j. \qquad (3)$$

The last step follows from the Binomial Theorem. Note that the terms $\left(-(l-1)i\right)^j$ for $j = 0, 1, \ldots$, etc., form a geometric progression as the ratio of two consecutive terms is always $-(l-1)i$.

Recall that a positive integer x, the *falling factorial* is defined as

$$(x)_n = x(x-1)(x-2)\ldots(x-(n-1)) = \sum_{k=0}^{n-1}(x-k),$$

for a positive integer n, and $(x)_0 = 1$. Note that $\frac{(x)_n}{n!} = \binom{x}{n}$.

Also, the *q-shifted factorial* is defined as

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right) = (1-a)(1-aq)\left(1-aq^2\right)\ldots\left(1-aq^{n-1}\right).$$

We say that a series of the form $\sum_{n=0}^{+\infty} a_n x^n$ is a *basic hypergeometric series* if $\frac{a_{n+1}}{a_n} = \frac{p(q)}{r(q)}$, for every n, where, s $p(q)$ and $r(q)$ are polynomials of arbitrary degree in q, q is a parameter such that $|q| < 1$, and $|x| < 1$, assuring convergence of the series.

We use the concepts and notation of basic hypergeometric series (see [2]) throughout this note. We have

$$(1-x)(1-xq)\ldots\left(1-xq^{n-1}\right) = (x:q)_n.$$

Note that

$$\lim_{n\to\infty}(x:q)_n = \prod_{n=0}^{\infty}\left(1-xq^n\right).$$

We see that for large values of k, the expression from (3)

$$L \cdot \sum_{j=0}^{k} \binom{k}{j} \left(-(l-1)i\right)^{j}$$

representing the outstanding loan balance at time k, can reach the value of 0, resulting in full repayment of the loan, provided that $1 < l < 2$ and $0 < i < 1$ (a very natural assumption for interest rates).

II. Varying Interest Rates

A more complicated scenario occurs if we instead use a varying interest rate. Let us assume that the interest rate (*forward rate,* or *short rate*) from time $j - 1$ to time j is $f_j = i^j$. This assumption is a special model for persistently falling interest rates, a scenario akin to the recent experience of developed economies such as Japan, or Germany, and, to a degree, the United States. We then have the following

Proposition 1. *Suppose a loan L' is to be repaid at l times the forward rate (or short rate) from time j – 1 to time j* given as $f_j = i^j$. *Then the loan will need either a single final payment (a balloon payment) or an additional annuity of scheduled payments, written as A, to complete the loan repayment, since*

$$OB_k = L'\left(1-(l-1)i\right)\left(1-(l-1)i^2\right)\ldots\left(1-(l-1)i^k\right) = \\ = \left((l-1)i;i\right)_k . \tag{4}$$

Note that the proposition is stated to imply that we require

$$L' = L'\left((l-1)i;i\right)_k + A,$$

if the original term of the payment structure was k periods.

We note that (4) is a special case of q-binomial formula [2, p. 5, eq. (6.23)]:

$$(x;q)_n = \sum_{j\geq 0} \frac{(q)_n (-x)^j q^{\binom{j}{2}}}{(q)_j (q)_{n-j}}. \tag{5}$$

Additionally, the left-hand side reduces to $(1-x)^n$ when $q=1$, which may be interpreted to be the case illustrated for (3).

Proof of the Proposition 1. In order to prove our claim, we need to show that for arbitrarily large k, or equivalently for $k \to \infty$, $OB_k > 0$. To see this, we use the limiting case of the q-binomial formula (1.5):

$$(xq;q)_\infty = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\frac{n(n+1)}{2}}}{(q)_n},$$

and select $x = l - 1$, $1 < l < 2$, and $q = i$. The terms in the series are clearly declining, tending to zero, and alternating from the first term of 1, which shows that the series converges to a value in the open interval (0, 1). Hence

$$L'\left((l-1)i;i\right)_\infty < L'$$

and

$$A = L'\left(1-\left((l-1)i;i\right)_\infty\right).$$

We also note that in the case when $0 < l < 1$, the value of A is larger than in the case when $1 < l < 2$, as in this unusual arrangement at the end of k periods there is an outstanding balance greater than the original loan L', by (4). Clearly, this means a final balloon payment is required.

III. Final Comments

If A is selected as a final payment and satisfies $A > L'li$ for $1 < l < 2$, then this is the balloon payment in the traditional sense, as the first payment is the otherwise largest one in such a scenario. While it seems unlikely that the short rate f_j may decline as rapidly as i^j, we believe that our model is still feasible. It may be of interest to apply our model in scenarios where it is believed than short rates will decline consistently in the future. Our main objective is to highlight the appearance of basic hypergeometric progression within the scope of financial mathematics.

References

[1] Wai-Sum Chan and Yin-Kuen Tse, *Financial Mathematics for Actuaries,* World Scientific Pub. Co. Inc., Second Edition, 2017.

[2] N. J. Fine, *Basic Hypergeometric Series and Applications,* Math. Surveys 27, American Mathematical Society, Providence, Rhode Island, U.S.A., 1985 (revision from orignal 1980 edition).