On the stability of 1D discrete dynamical systems: applications to population dynamics

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***Abstract*—In this paper we present a survey in the theory of stability of one-dimensional discrete-time dynamical systems. The main goal is to write a document that may be used as a pedagogical instrument in stability analysis for researchers that are not familiar with these techniques. Hence, we present the principal definitions and results in this field and a collection of classical population models widely used in population ecology. The dynamics of these models is, in general, a time-parameter dependent and the idea is to describe what factors in the parameter affect population size and how and why a population changes over time. Moreover, the presented techniques may be extended for different fields of science since the addressed results and examples are standard and may be adapted for a particular situation.**

***Index Terms*—Local Stability, Global Stability, Population Dynamics, Applications**

I. Introduction

One-dimensional discrete models are an appropriate mathematical tool to model the behavior of populations with non-overlapping generations. This subject has been intensely investigated by different researchers and is part of the solid foundations of the modern theory of discrete dynamical systems.

A discrete dynamical system (or difference equation) is a relation governed by the rule

$$x_{n+1} = f_n(x_n), \quad n = 0,\ 1,\ 2,\ \ldots, \tag{1}$$

where $x \in X$ and X is a topological space. Here, the orbit of a point x_0 is generated by the composition of the sequence of maps

$$f_0, f_1, f_2, \ldots.$$

Explicitly,

$$\begin{aligned} x_1 &= f_0(x_0), \\ x_2 &= f_1(x_1) = f_1 \circ f_0(x_0), \\ &\vdots \\ x_{n+1} &= f_n \circ f_{n-1} \circ \ldots \circ f_1 \circ f_0(x_0), \\ &\vdots \end{aligned}$$

If $f_0 = f_1 = f_2 = \ldots$, then the equation is said to be autonomous, otherwise it is non-autonomous. If the sequence of maps is periodic, i.e., $f_{n+p} = f_n$, for all $n = 0,\ 1,\ 2,\ \ldots$ and some positive integer $p > 1$, then we deal with non-autonomous periodic difference equations. Systems where the sequence of maps is periodic, model population with fluctuation habitat, and they are commonly called periodically forced systems.

Notice that the non-autonomous periodic difference equation (1) does not generate a discrete (semi)dynamical system [6] as it may not satisfy the (semi)group property. One of the most effective ways of converting the non-autonomous difference equation (1) into a genuine discrete (semi)dynamical system is the construction of the associated skew-product system as described in a series of papers by Elaydi and Sacker [6]–[8], [10]. It is noteworthy to mention that this idea was originally used to study non-autonomous differential equations by Sacker and Sell [22].

An ordered set of points $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ is an $r-$periodic cycle in X if

$$f_{(i+nr) \bmod p}(\overline{x}_i) = \overline{x}_{(i+1) \bmod r},\ n = 0,\ 1,\ 2,\ \ldots.$$

In particular,

$$f_i(\overline{x}_i) = \overline{x}_{i+1},\ 0 \le i \le r-2,$$

and

$$f_t(\overline{x}_{t \bmod r}) = \overline{x}_{(t+1) \bmod r},\ r-1 \le t \le p-1.$$

It should be noted that the $r-$periodic cycle C_r in X generates an $s-$periodic cycle on the skew-product $X \times Y$ ($Y = \{f_0, f_1, \ldots, f_{p-1}\}$) of the form

$$\widehat{C}_s = \{(\overline{x}_0, f_0), (\overline{x}_1, f_1), ..., (\overline{x}_{(s-1) \bmod r}, f_{(s-1) \bmod p})\},$$

where $s = lcm[r, p]$ is the least common multiple of r and p.

To distinguish these two cycles, the $r-$periodic cycle C_r on X is called an $r-$geometric cycle (or simply $r-$periodic cycle when there is no confusion), and the $s-$periodic cycle $\widehat{C}_s$ on $X \times Y$ is called an $s-$complete cycle. Notice that either $r < p$, or $r = p$ or $r > p$.

Define the composition operator Φ as follows

$$\Phi_n^i = f_{n+i-1} \circ \ldots \circ f_{i+1} \circ f_i.$$

When $i = 0$ we write Φ_n^0 as Φ_n.

This work was partially supported by FCT/Portugal through project PEst-OE/EEI/LA0009/2013.

As a consequence of the above remarks it follows that the $s-$complete cycle $\widehat{C}_s$ is a fixed point of the composition operator Φ_s^i. In other words we have that

$$\Phi_s^i(\overline{x}_{i \bmod r}) = \overline{x}_{i \bmod r}.$$

If the sequence of maps $f_i,\ i = 0,\ 1,\ 2,\ \ldots$ is a parameter family of maps one-to-one in the parameter, then by [11] we have that $\overline{x}_{i \bmod p}$ is a fixed point of Φ_p.

Before ending this short introduction, we mention that in section II we present the principal results concerning the stability of fixed points in autonomous discrete dynamical systems. The results presented in this section may be extended to periodic systems substituting the map f for the composition operator Φ. The next two sections are devoted to applications. In section III we present the principal models for one-dimensional population dynamics. In the next section we refer some studies in non-autonomous periodic equations.

Finally, this survey may be used as a pedagogical instrument in stability analysis for students or researchers that are not familiar with these techniques and aims to study discrete dynamical systems. Moreover, it may be used for researchers in other fields that are familiar with some basic tools in Analysis. The presented examples in population dynamics are known and may be extended for other areas following the exposed techniques.

II. General results

Consider an interval $I \subseteq \mathbb{R}$ and an autonomous map $f : I \to I$. A point $x^* \in \mathbb{R}$ is said to be a fixed point (or equilibrium point) of f if $f(x^*) = x^*$, and given $x_0 \in \mathbb{R}$, we define its orbit $O(x_0)$ as the set of points

$$O(x_0) = \{x_0,\ f(x_0),\ f^2(x_0),\ f^3(x_0),\ \ldots\},$$

where $f^n = f \circ f^{n-1}$, for $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers and $\circ$ represents the composition of functions.

One of the main objectives of the stability theory of discrete dynamical systems is to study the behavior of orbits when the starting points are near fixed points.

Let $f : I \to I$ be a map and x^* be a fixed point of f, where I is an interval of real numbers. Then:

1) The fixed point x^* is said to be locally stable if, for any $\epsilon > 0$, there exits $\delta > 0$ such that, for all $x_0 \in I$ with $|x_0 - x^*| < \delta$, we have $|f^n(x_0) - x^*| < \epsilon$, for all $n \in \mathbb{N}$. Otherwise, the fixed point x^* will be called unstable.
2) The fixed point x^* is said to be attracting if there exists $\eta > 0$ such that $|x_0 - x^*| < \eta$ implies $\lim\limits_{n\to\infty} f^n(x_0) = x^*$.
3) The fixed point x^* is said locally asymptotically stable if it is both stable and attracting. If in the previous item $\eta = \infty$, then x^* is said to be globally asymptotically stable.

Fig. 1 illustrates the idea behind the definition of stability.

Working with concrete examples, the definition of stability may not be the most practical tool to show the stability of a fixed point. There exists a simple but powerful criterion for knowing the local stability of fixed points. We may divide the

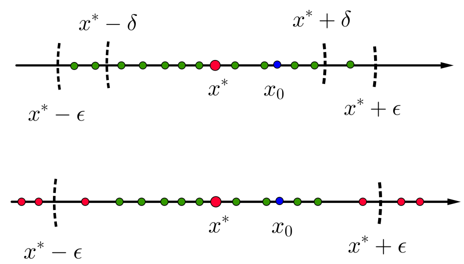

Fig. 1. Stable vs unstable fixed point. In the first case, the orbit of a starting point x_0 in a neighborhood δ of x^* stay in a neighborhood ϵ of x^* while in the second case, after certain order, the orbit start to be out of a neighborhood ϵ of x^*.

fixed point into two categories: hyperbolic and nonhyperbolic. A fixed point x^* of a map f is said to be hyperbolic if $|f'(x^*)| \neq 1$, where f' denotes the derivative of the function f. Otherwise, the fixed point is nonhyperbolic.

Theorem(Elaydi [4], page 25):

Let x^* be a hyperbolic fixed point of a map f, where f is continuous and differentiable at x^*. The following statements hold true:

1) If $|f'(x^*)| < 1$, then x^* is locally asymptotically stable.
2) If $|f'(x^*)| > 1$, then x^* is unstable.

The stability criteria for nonhyperbolic fixed points are more complex and are summarized in the following theorem. Before presenting it we need to introduce the notion of Schwarzian derivative.

The Schwarzian derivative, Sf, of a function f, is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

In particular, when $f'(x^*) = -1$, we have $Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2$.

Theorem(Elaydi [4], pages 28-30): Let x^* be a fixed point of a map f and f', f'' and f''' be continuous at x^*.

1) Let $f'(x^*) = 1$.
   a) If $f''(x^*) > 0$, then x^* is unstable but semi-stable from the left.
   b) If $f''(x^*) < 0$, then x^* is unstable but semi-stable from the right.
   c) If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.
   d) If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is locally asymptotically stable.
2) Let $f'(x^*) = -1$.
   a) If $Sf(x^*) < 0$, then x^* is locally asymptotically stable.
   b) If $Sf(x^*) > 0$, then x^* is unstable.

In applications it is important to know whenever the conditions of local stability imply global stability. The precedent results gives conditions on local stability. In 1955 W. Coppel stated the following result:

Theorem(Coppel [1]): Let $I = [a, b] \subseteq \mathbb{R}$ and $f : I \to I$ be a continuous map. If the equation $f(f(x)) = x$ has no roots, with the possible exception of the roots of the equation $f(x) = x$, then every orbit under the map f converges to a fixed point.

Coppel's theorem is not enough to ensure global stability. It is necessary to have uniqueness of the fixed point. In certain cases, depending the model we can establish global stability since only a unique fixed point will play a rule, as we will illustrate in the next sections.

III. Autonomous models

In this section we apply the results stated in the previous section to some important autonomous models in applications. We will start by the Ricker model which is a fishing model.

Example I - Ricker model: The Ricker model is given by

$$x_{n+1} = x_n e^{p - x_n},$$

where $x_n \geq 0$ is the density of the population at the period of time n and $p > 0$ is the carrying capacity of the population.

The map of the model is given by $f(x) = xe^{p-x}$. There are two fixed points, namely $x^* = 0$ and $x^* = p$. The origin is an unstable fixed point provide that $f'(0) = e^p > 1$. The positive fixed point is locally asymptotically stable whenever $0 < p \leq 2$ and unstable when $p > 2$. Notice that $|f'(p)| < 1$ implies that $0 < p < 2$ and since $f'(2) = -1$ we have $Sf(2) = -1 < 0$.

Now, solving the equation $f(f(x)) = x$, one can show that there are only two solutions whenever $0 < p \leq 2$, the origin and $x = p$, precisely the fixed pints of the map f. Since the origin is unstable, and $x^* = p$ is the unique fixed point in the positive real line, we have that the conditions of local stability of $x^* = p$ will implies global stability with respect to the positive real line. This means that every orbit starting at $x_0 > 0$ will converge to $x^* = p$ whenever $0 < p \leq 2$.

In Figure 2 is presented a cobweb diagram for this model. Notice that, a cobwebbing diagram is a geometrical toll where one can see the location of the values in the orbit of a starting point x_0. These values are located in the diagonal line $y = x$. In this case, it is clear that the orbit of $x_0 = 0.1$ converge to $x^* = 1.73$.

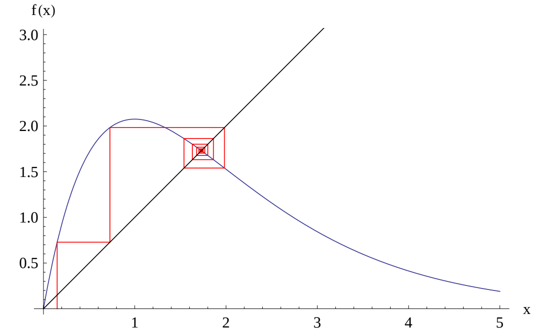

Fig. 2. Cobweb diagram for the Ricker model $x_{n+1} = x_n e^{p-x_n}$ when $p = 1.73$ and the starting point $x_0 = 0.1$. This example illustrates that the origin is an unstable fixed point and the positive fixed point is globally asymptotically stable.

It follows a difference equation which is the discrete version of the Verhulst model [24], [25] also called logistic map, a population model well known and studied in this field. We recall that the dynamics of this equation has played a paramount importance and it is present in the foundations and in the development of the modern theory of discrete dynamical systems.

Example II - Logistic model: The 1D logistic equation is given by

$$x_{n+1} = \mu x_n(1 - x_n),$$

where $x_n \in [0,\ 1]$ is the density and $\mu \in (0, 4)$ is a control parameter that represents a combined rate for reproduction and starvation.

We remark that this equation is found to be the most suitable model for the study of the surplus production of the population biomass of species in the presence of limiting factors such as food supply or disease. The above logistic model can possess stable, unstable, periodic and chaotic behaviors and thus receives wide attention due to the great implications of it in chaos theory (see May [20] for details at this point).

Since the map is given by $f(x) = \mu x(1 - x)$, the model has two fixed points, the origin and $x^* = \frac{\mu - 1}{\mu}$.

From the relation $f'(0) = \mu$ we have that the origin is locally asymptotically stable when $0 < \mu < 1$ and unstable when $\mu > 1$. When $\mu = 1$ we have $f'(0) = 1$. It follows that $f''(0) = -2 < 0$ and thus the origin is semi-stable from the right.

It is a straightforward computation to see that $x^* = \frac{\mu - 1}{\mu}$ is locally asymptotically stable whenever $1 < \mu \leq 3$ and unstable when $\mu \in (0, 1] \cup (3, 4)$.

Following a similar idea as the precedent example, one can show that the solutions of the equation $f(f(x)) = x$ are $x = 0$ and $x^* = \frac{\mu - 1}{\mu}$ whenever $0 < \mu \leq 3$. There are two cases: (i) $x^* = 0$ is globally asymptotically stable when $0 < \mu \leq 1$ provide that it is the unique fixed point of f in $[0, 1]$ and (ii) when $1 < \mu \leq 3$ the fixed point $x^* = \frac{\mu - 1}{\mu}$ is globally asymptotically stable with respect to the interior of the unit interval since it is the unique fixed point in this region.

Example III - Beverton-Holt model: The 1D Beverton-Holt map is given by

$$f(x) = \frac{rKx}{K + (r - 1)x},$$

where $x \geq 0$ is the density, $K > 0$ is the carrying capacity and $r > 0$ is the growth rate of the population. There are two fixed points, the origin which is locally asymptotically stable when $0 < r \leq 1$ and a positive fixed point $x^* = K$ which is locally asymptotically stable whenever $r > 1$.

In this example we do not need to apply Coppel's theorem to establish global stability since the model is monotone. Hence, the origin is globally asymptotically stable with respect to the interval $[0, K)$ whenever $0 < r \leq 1$, and $x^* = K$ is globally asymptotically stable with respect to the positive real line whenever $r > 1$.

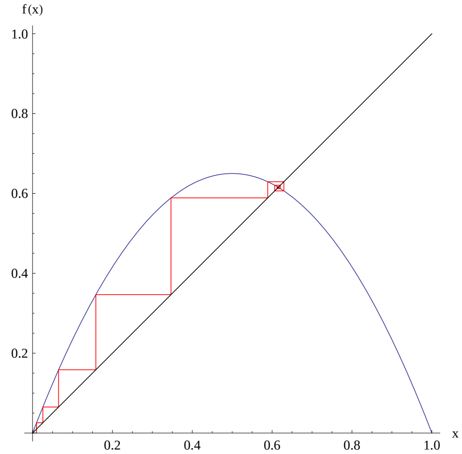

Fig. 3. Cobweb diagram for the logistic map when $x_0 = 0.15$ and $\mu = 2.6$. An orbit of starting point in the interior of the unit interval converges to the positive fixed point since it is globally stable in this set.

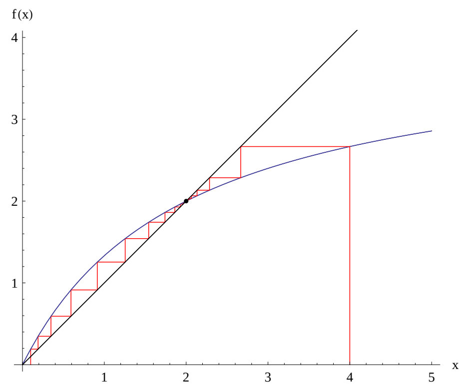

Fig. 4. Cobweb diagram for the Beverton-Holt model when $r = 2$ and $K = 2$. In this case we present the orbit of two initial points. It illustrates the unstability of the origin and the globally stability of the positive fixed point.

Example IV - Ricker with Allee effect: The modified Ricker model with Allee effect is given by

$$x_{n+1} = x_n^2 e^{p-x_n},$$

where $x_n \geq 0$ is the density of the population and $p > 0$ is the carrying capacity.

The fixed points of the model are the solutions of the equation $x^2 e^{p-x} = x$. From this relation it follows $x^* = 0$ and $xe^{p-x} = 1$. This last equation has no solution if $p < 1$, exactly one solution $x^* = 1$ if $p = 1$ and two solution, $x^* = \mathbf{A} < 1$ and $x^* = \mathbf{K} > 1$ if $p > 1$. In population dynamics these last fixed points are known as threshold point ($\mathbf{A}$) and carrying capacity ($\mathbf{K}$).

Hence, there are 3 cases to consider:

(i) $p < 1$. In this case the origin is a globally asymptotically stable fixed point provide that it is the unique fixed point in the non-negative real line. Notice that $f'(0) = 0$.

(ii) $p = 1$. There are two fixed points in the model, the origin and $x^* = 1$. The origin is locally asymptotically stable since $f'(0) = 0$ and its basin of attraction[1] is the set $[0,\ 1[\cup]\mathbf{A_r},\ +\infty[$, where $\mathbf{A_r}$ is the right pre-image of 1, i.e., the greatest solution of the equation $x^2 e^{1-x} = 1$ which is in this case ≈ 3.51286. The fixed point $x^* = 1$ is semistable from the right since $f'(1) = 1$ and $f''(1) = -1 < 0$. Its basin of attraction is the set $[1,\ \mathbf{A_r}]$.

(iii) $p > 1$. There are three fixed points, the origin, $x^* = \mathbf{A} < 1$ and $x^* = \mathbf{K} > 1$. The origin is locally asymptotically stable fixed point and its basin of attraction is the set $[0, \mathbf{A_r}[\cup]\mathbf{A_r},\ +\infty[$, where $\mathbf{A_r}$ is the right pre-image of $\mathbf{A}$.

In order to determine the stability of $\mathbf{A}$ and $\mathbf{K}$ notice that

$$f'(x) = x(2-x)e^{p-x},$$

and for the non-trivial values we have

$$f'(x) = \frac{(2-x)}{x} f(x).$$

Hence $|f'(\mathbf{A})| = |2 - \mathbf{A}|$ and $|f'(\mathbf{K})| = |2 - \mathbf{K}|$. Since $0 < \mathbf{A} < 1$ and $\mathbf{K} > 1$ we have that $\mathbf{A}$ is an unstable fixed point whereas $\mathbf{K}$ is locally asymptotically stable whenever $1 < \mathbf{K} < 3$. If this is the case, then its basin of attraction is $]\mathbf{A}, \mathbf{A_r}[$.

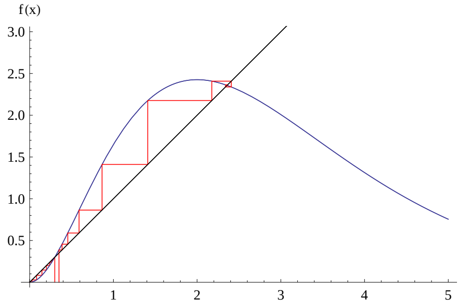

Fig. 5. Cobweb diagram of the Ricker model with Allee effect when the parameter $p = 2$. It illustrates the local stability of the origin and the carrying capacity and the instability of the threshold point.

Example V - Polynomial with Allee effect: Let us consider the difference equation given by

$$x_{n+1} = \mu_n x_n^{k_n} \left(1 - x_n\right), \tag{2}$$

where $x_n \in [0, 1]$, $\mu_n > 0$ and $k_n = 2, 3, 4, \ldots$ for all non negative integer n. For more details about this equation please see [18].

Equation (2) may be represented by the map

$$f_n(x) = \mu_n x^{k_n} \left(1 - x\right).$$

Notice that when $\mu_n = \mu$ and $k_n = 1$ for all n, Equation (2) is the logistic equation studied in Example III.

In order to insure that $x_n \in I = [0, 1]$ for all n, we make the following assumption concerning the parameters

[1]The basin of attraction (or the stable set) of a fixed point consists of all points that are forward asymptotic to it.

H: $\mu_n \leq \left(\dfrac{k_n+1}{k_n}\right)^{k_n}(k_n+1),\ n=0,1,2\ldots.$

Assumption **H** guarantees that all the orbits in (2) are bounded. Furthermore, it guarantees that f_n maps the interval I into the interval I for all $n=0,1,2\ldots.$

Let us now study the dynamics of the particular map $f(x)=\mu x^k(1-x)$, with $x\in I$, $\mu>0$ and $k=2,3,\ldots$. To find the fixed points of f we determine the solutions of the equation $\mu x^k(1-x)=x$. After eliminating the trivial solution, $x=0$, the positive fixed points are the solutions of

$$\mu x^{k-1}(1-x)=1, \tag{3}$$

or equivalently

$$\ln(\mu)=-(k-1)\ln x-\ln(1-x). \tag{4}$$

Letting $g(x)=-(k-1)\ln x-\ln(1-x)$, we see that $g(x)>0$ for all $x\in(0,1)$. Moreover, g is convex in the unit interval since $g'(x)>0$, for all $x\in I$, and attains its minimum at $g(c_g)$ where $c_g=\frac{k-1}{k}$ is the unique critical point of g in the unit interval. Let $\mathbf{O}_\mu$ be the immediate basin of attraction of the origin.

1) If $g(c_g)>\ln(\mu)$, then Eq. (4) has no solution. Hence, $x^*=0$ is the unique fixed point of the map f whenever $\mu<k\left(\frac{k}{k-1}\right)^{k-1}$. Under this scenario $x^*=0$ is globally asymptotically stable, given that it is the unique fixed point in I. Notice that at the origin we have $f'(0)=0$ and that $\mathbf{O}_\mu=[0,1]$.
2) If $g(c_g)=\ln(\mu)$, then Eq. (4) has a unique solution, $x^*=\frac{k-1}{k}=c_g$. Hence, the map f has a unique positive fixed point when $\mu=k\left(\frac{k}{k-1}\right)^{k-1}$. In this case and using (3), we obtain $|f'(x^*)|=1$ and $|f''(x^*)|=-k^2<0$, that allows us to conclude that x^* is an unstable fixed point, but semi-stable from the right. Moreover, its immediate basin of attraction is the set $\left[x^*,\max f^{-1}(\{x^*\})\right]$ where $f^{-1}(\{x^*\})$ is the pre-image of $\{x^*\}$. Notice that $\mathbf{O}_\mu=I\setminus\left[x^*,\max f^{-1}(\{x^*\})\right]$.
3) If $g(c_g)<\ln(\mu)$, then Eq. (4) has two positive solutions. Hence, the map f possesses two positive fixed points whenever $\mu>k\left(\frac{k}{k-1}\right)^{k-1}$. The smaller, denoted as $\mathbf{A}_\mu$, is known as a threshold point and the greater, denoted by $\mathbf{K}_\mu$, is known as a carrying capacity. Under this scenario, the fixed point $\mathbf{A}_\mu$ is always unstable and the fixed point $\mathbf{K}_\mu$ is locally asymptotically stable in the interval $\left(\mathbf{A}_\mu,\max f^{-1}(\{\mathbf{A}_\mu\}\right)$ if $\left|k-\mu\mathbf{K}_\mu^k\right|<1$. Moreover, $\mathbf{O}_\mu=[0,\mathbf{A}_\mu)\cup\left(\max f^{-1}(\{\mathbf{A}_\mu\}),1\right]$.

Notice that the sequence $a_k=\left(\dfrac{k+1}{k}\right)^k(k+1)$ that is used to define Assumption **H** is increasing for $k=2,3,\ldots$. We now resume the precedent ideas in the following result, for a general integer $k=2,3,\ldots$:

Theorem: Let $f(x)=\mu x^k(1-x)$, $k=2,3,\ldots$. Then the following yields:

1) If $\mu<k\left(\frac{k}{k-1}\right)^{k-1}$, then $x^*=0$ is a globally asymptotically stable fixed point of f and its basin of attraction is the unit interval.
2) If $\mu=k\left(\frac{k}{k-1}\right)^{k-1}$, then the map has two fixed points, the origin and a positive fixed point $x^*=\frac{k-1}{k}$. This last one is locally asymptotically stable from the right and its immediate basin of attraction is the set $\left[x^*,\max f^{-1}(\{x^*\})\right]$. Moreover, $\mathbf{O}_\mu=I\setminus\left[x^*,\max f^{-1}(\{x^*\})\right]$.
3) If $\mu>k\left(\frac{k}{k-1}\right)^{k-1}$, then the map has three fixed points, the origin, a threshold fixed point $\mathbf{A}_\mu$ and a carrying capacity $\mathbf{K}_\mu$ such that $\mathbf{A}_\mu<\mathbf{K}_\mu$. The threshold fixed point is always unstable and if $|k-\mu\mathbf{K}_\mu^k|<1$ the carrying capacity is locally asymptotically stable with a basin of attraction given by the set $\left(\mathbf{A}_\mu,\max f^{-1}(\{\mathbf{A}_\mu\})\right)$. Moreover, $\mathbf{O}_\mu=I\setminus\left[\mathbf{A}_\mu,\max f^{-1}(\{\mathbf{A}_\mu\})\right]$.

Remark: Before ending this example let us have a particular look in the dynamics of the autonomous equation when $k=2$, i.e., the dynamics of the modified logistic equation with Allee effect when the map is given by $f(x)=\mu x^2(1-x)$.

1) If $\mu<4$, then the origin is a globally asymptotically stable fixed point provided that it is the unique fixed point in the unit interval.
2) If $\mu=4$, then the map possesses two fixed points, the origin and $x^*=\frac{1}{2}$. The basin of attraction of the origin is
$$\mathbf{O}_4=\left[0,\frac{1}{2}\right)\cup\left(\frac{1+\sqrt{5}}{4},1\right], \tag{5}$$
while the basin of attraction of the positive fixed point is $\left[\frac{1}{2},\frac{1+\sqrt{5}}{4}\right]$. Notice that $x^*=\frac{1}{2}$ is a fixed point semi-stable from the right.
3) If $4<\mu$, then the map has three fixed points, the origin, the threshold point $\mathbf{A}_\mu=\frac{1}{2}\left(1-\sqrt{\frac{\mu-4}{\mu}}\right)$ and the carrying capacity $\mathbf{K}_\mu=\frac{1}{2}\left(1+\sqrt{\frac{\mu-4}{\mu}}\right)$.
It is a straightforward computation to see that, when $\mu>4$,
$$|f'(\mathbf{A}_\mu)|=3+\frac{\mu}{2}\left(-1+\sqrt{\frac{\mu-4}{\mu}}\right)>1.$$
Hence, the fixed point $\mathbf{A}_\mu$ is unstable.
Similarly, we see that
$$|f'(\mathbf{K}_\mu)|=\left|3-\frac{\mu}{2}\left(1+\sqrt{\frac{\mu-4}{\mu}}\right)\right|<1 \text{ iff } 4<\mu<\frac{16}{3}.$$
When $\mu=\frac{16}{3}$ we have $f'(\mathbf{K}_\mu)=-1$. Forward computations show that the Schwarzian derivative evaluated at the fixed point is negative, i.e., $Sf(\mathbf{K}_\mu)<0$. It follows that the fixed point $\mathbf{K}_\mu$ is asymptotically stable. Thus, the fixed point $x^*=\mathbf{K}_\mu$ is locally asymptotically stable whenever $4<\mu\leq\frac{16}{3}$ and its basin of attraction is the set $\left(\mathbf{A}_\mu,\ \max f^{-1}(\{\mathbf{A}_\mu\})\right)$. Moreover,
$$\mathbf{O}_\mu=[0,\mathbf{A}_\mu)\cup\left(\max f^{-1}(\{\mathbf{A}_\mu\}),1\right]. \tag{6}$$

IV. NON-AUTONOMOUS MODELS

In this section we present some studies for particular periodic difference models. We notice that the study of this kind of equations is quite complicate and in certain cases it is not possible to find explicitly the fixed points due the complexity of computations, specially nontrivial fixed points.

Example VI - Periodic Ricker map:

Let us consider the periodic difference equation given by the following equation

$$x_{n+1} = R_n(x_n),$$

where the sequence of maps $R_n(x)$ is given by

$$R_n(x) = xe^{r_n - x}, \ n = 0, \ 1, \ 2 \ldots, \tag{7}$$

$x \geq 0$ is the density of the population and $r_n > 0$, $n = 0, \ 1, \ 2 \ldots$ is the sequence of individual carrying capacities.

Notice that the local stability condition for each individual map $R_i(x)$ is given by

$$0 < r_i \leq 2, \quad i = 0, \ 1, \ 2 \ldots,$$

as is shown in Example III.

In order to have periodicity we require that $R_{n+p} = R_n$, for all $n = 0, 1, 2, \ldots$, i.e., the sequence of parameters satisfies $r_n = r_{n \bmod p}$, for all n. It is clear that the composition map

$$\Phi_p(x) = R_{p-1} \circ \ldots \circ R_1 \circ R_0(x)$$

is continuous in $\mathbb{R}_0^+$.

In [21] R. Sacker showed that the map Φ_p has a globally asymptotically stable fixed point whenever the periodic sequence of parameters satisfies $0 < r_n \leq 2$, $n = 0, 1, 2, \ldots$. Since the sequence of maps is one-to-one relative to the parameters, it follows from [11] that the globally asymptotically stable fixed point of Φ_p generates a globally asymptotically stable $p-$periodic cycle of the form

$$\{\overline{x}_0, \ \overline{x}_1, \ \ldots, \ \overline{x}_{p-1}\}.$$

Using the chain rule of derivative it follows that

$$\Phi_p'(\overline{x}_0) = R_{p-1}'(\overline{x}_{p-1}) R_{p-2}'(\overline{x}_{p-2}) \ldots R_1'(\overline{x}_1) R_0'(\overline{x}_0).$$

Since $R_i'(x) = (1-x)e^{p_i - x}$ and the dynamics of the periodic orbit is $\overline{x}_{i+1} = \overline{x}_i e^{r_i - \overline{x}_i}$, $i = 0, \ 1, \ 2 \ \ldots, \ p-1$, the stability condition of the periodic orbit is

$$\prod_{i=0}^{p-1} |1 - \overline{x}_i| < 1. \tag{8}$$

Later on, Elaydi et al. [12] noticed that the region of stability in the parameter space determined by Sacker may be larger as it is shown in Fig. 6 for a 2−periodic equation. They have been determined the boundary of the region and in a recent paper, Liz [19] showed global stability in this region using the following result:

Theorem(Corollary 2.9 in [13] by El-Morshedy & López): Let $a \geq 0$, $b > a$ and $g : (a, \ b) \to [a, \ b]$ be a continuous map with a unique fixed point x^* such that $(g(x) - x)(x - x^*) < 0$

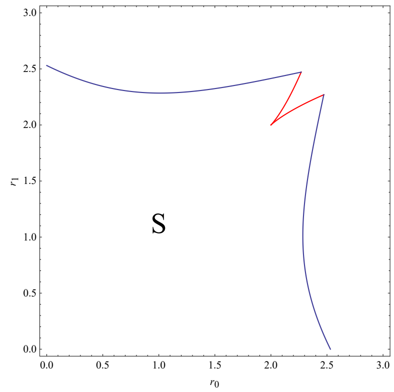

Fig. 6. Region S where the 2−periodic Ricker equation has a globally asymptotically stable 2−periodic cycle. The curves are part of the region of global stability. Once the parameters crosses these curves the 2−periodic cycle becomes unstable.

for all $x \neq x^*$. Assume that there are points $a \leq c < x^* < d \leq b$ such that the restriction of g to $(c, \ d)$ has at most one turning point and whenever it makes sense, $g(x) \leq g(c)$ for every $x \leq c$, and $g(x) \geq g(d)$ for every $x \geq d$. If g is decreasing at x^*, assume additionally that $Sg(x) < 0$ for all $x \in (c, \ d)$ except at most one critical point og g and $-1 < g'(x^*) < 0$. Then the fixed point x^* is globally stable.

It remains as an open problem to show global stability for $p \geq 3$.

Example VII - Periodic Beverton-Holt model:

Let $x_{n+1} = B_n(x_n)$, $n = 0, \ 1, \ 2, \ldots$ where the map B_n is given by

$$B_n(x) = \frac{rK_n x}{K_n + (r-1)x}. \tag{9}$$

Here $x \geq 0$ is the density, the parameter $r > 1$ is the grow rate and the sequence of parameters $K_n > 0$ are the carrying capacities of each individual population. In Example III is established the conditions for stability of each individual map B_n.

Let us now assume that $K_{n+p} = K_n$ for all n and $p > 1$, i.e., the sequence of maps B_n is $p-$periodic. Since each individual map is monotone and the composition of monotone maps is monotone, we have that Φ_p is a monotone map. Moreover, the orbits are bounded since $B_n(x) < \frac{r}{r-1}K_n$ for all n.

It follows from the Brouwer's fixed point theorem that Φ_p has a fixed point. Due the monotonicity we have that the fixed point is globally asymptotically stable. This fixed point of Φ_p generates a globally asymptotically stable $p-$periodic cycle in the original equation (9) of the form

$$\{\overline{x}_0, \ \overline{x}_1, \ \ldots, \ \overline{x}_{p-1}\}.$$

In a famous conjecture, Chushing and Hensen [2], [3] stated that the average of the individual carrying capacities is less than the average of the numbers in the $p-$periodic cycle, i.e.,

$$\frac{K_0 + K_1 + \ldots + K_{p-1}}{p} < \frac{\overline{x}_0 + \overline{x}_1 + \ldots + \overline{x}_{p-1}}{p}.$$

Using Jensen's inequality some researchers solved positively this conjecture. To cite few [5]–[10], [15], [16], [23].

In conclusion, forcing the system may be beneficial for the population since the carrying capacity of the periodic population will be greater than the individual carrying capacities.

Example VIII - Generalized periodic logistic:

We start this example presenting a result related to the non-autonomous equation (2) when $k = 2$ (although it may be extended for other values of the parameter k as well). It is not hard to prove the following:

Lemma: Consider the non-autonomous difference equation given by

$$x_{n+1} = \mu_n x_n^2 \left(1 - x_n\right), \tag{10}$$

where $x_n \in [0,1]$, $\mu_n \in \left(0, \frac{27}{4}\right]$, for $n = 0, 1, 2 \ldots$, and $\mathbf{O}_\mu$ the immediate basin of attraction of the origin. Then

$$4 \leq \mu_1 \leq \mu_2 \leq \frac{27}{4} \Rightarrow \mathbf{O}_4 \supseteq \mathbf{O}_{\mu_1} \supseteq \mathbf{O}_{\mu_2} \supseteq \mathbf{O}_{\frac{27}{4}}, \tag{11}$$

where $\mathbf{O}_4$ is given by (5) and

$$\mathbf{O}_{\frac{27}{4}} = \left[0, \frac{9-\sqrt{33}}{18}\right) \cup \left(\max f^{-1}\left(\left\{\mathbf{A}_{\frac{27}{4}}\right\}\right), 1\right], \tag{12}$$

where $\max f^{-1}\left(\left\{\mathbf{A}_{\frac{27}{4}}\right\}\right) \approx 0.971\,62$.

Let us now turn our attention to the non-autonomous periodic equation (2). We will study the case where the sequence of maps is $p-$periodic, *i.e.*, when $f_{n+p} = f_n$, for all $n = 0, 1, 2, \ldots$. Under this scenario, equation (2) is $p-$periodic.

The dynamics of the non-autonomous $p-$periodic equation (2) is completely determined by the following composition operator

$$\Phi_p = f_{p-1} \circ \ldots \circ f_1 \circ f_0.$$

From assumption $\mathbf{H}$ it follows that $\Phi_p(I) \subseteq I$ with $\Phi_p(0) = 0$ and $\Phi_p(1) = 0$. Hence, by the Brouwer's fixed point theorem [14], the composition operator Φ_p has a fixed point in the unit interval.

It is clear that $x^* = 0$ is a locally asymptotically stable fixed point of Φ_p provided that $|\Phi_p'(0)| = 0$. Now, if $\Phi_p(x) < x$, for all $x \in (0,1)$, then $x^* = 0$ is the unique fixed point of the composition operator Φ_p in the unit interval. In this case, $x^* = 0$ is a globally asymptotically stable fixed point and its basin of attraction is the entire unit interval. This is the case where local stability implies global stability in the sense that every orbit of $x_0 \in I$ converge to the origin.

Notice that, if C_{Φ_p} is the set of critical points of Φ_p, i.e., if C_{Φ_p} contains all the solutions in the unit interval of the p equations $\Phi_i(x) = c_i$, $i = 0, 1, \ldots, p-1$, where c_i is the critical point of the map f_i, then $\Phi_p(x) < x$, for all $x \in (0,1)$ if $\Phi_p(c_{\Phi_p}) < c_{\Phi_p}$, where $c_{\Phi_p} \in C_{\Phi_p}$.

Now, if $|\Phi_p(x)| > x$ for some $x \in (0,1)$, the composition operator Φ_p has more than one fixed point. We know from Coppel's Theorem [1] that every orbit converges to a fixed point if and only if the equation $\Phi_p \circ \Phi_p(x) = x$ has no solutions with the exception of the fixed points of Φ_p.

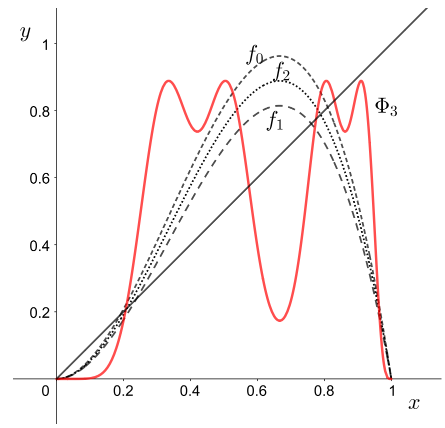

Fig. 7. Composition of three generalized logistic maps. The composition map Φ_3 is represented by the solid curve and the individual maps are represented by the dashed curves. The values of parameters are $k = 2$, $\mu_0 = 6.5$ (f_0), $\mu_1 = 5.5$ (f_1) and $\mu_2 = 6$ (f_2).

It is not possible, in general, to say much concerning the number of fixed points of Φ_p since we have many scenarios. However, if all maps f_i have a threshold fixed point $\mathbf{A}_i$ and we let $\mathbf{A}_m = \min\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{p-1}\}$ and $\mathbf{A}_M = \max\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{p-1}\}$, then one can show that the minimal positive fixed point of Φ_p, $\mathbf{A}_{\Phi_p}$, lies between $\mathbf{A}_m$ and $\mathbf{A}_M$ and is, in fact, an unstable fixed point. Under this scenario, the immediate basin of attraction of the origin is $\cup_{i \geq 1} J_i$ where $J_i \subset I$ and

$$\Phi_p(J_i) \subset [0, \mathbf{A}_{\Phi_p}).$$

See Fig. 7 for an example of this scenario.

We remark that each fixed point of the composition map Φ_p, with the exception of $x^* = 0$, generates a periodic orbit in equation (2). More precisely, if x^* is a non-trivial fixed point of Φ_p, then

$$\overline{C} = \{\overline{x}_0 = x^*, \overline{x}_1 = f_0(\overline{x}_0), \overline{x}_2 = f_1(\overline{x}_1), \ldots, \overline{x}_{p-1} = f_{p-2}(\overline{x}_{p-2})\}$$

is a periodic cycle of equation (2), which is locally asymptotically stable if

$$|\Phi_p'(x^*)| = \left|\prod_{i=0}^{p-1} f_i'(\overline{x}_i)\right| < 1.$$

Notice that, due the periodicity of the maps f_i, we have $\overline{x}_p = f_{p-1}(\overline{x}_{p-1}) = \overline{x}_0$, $\overline{x}_{p+1} = \overline{x}_1$, and so on.

From the dynamical point of view, it is interesting to know the region where the stability of the fixed points occurs. Since we are not able to find explicitly the fixed points of the composition map Φ_p for general values of the parameters k_i and μ_i, $i = 0, 1, \ldots, p-1$, we will particularize and study the cases where this is possible as are the cases when $p = 2, 3, 4$ and $k = 2$, i.e., we will study the dynamics of the system when the sequence of maps is $2-$periodic and given by

$$f_{n \ mod(2)}(x) = \mu_{n \ mod(2)} \ x^k(1-x), \quad k = 2, 3, 4.$$

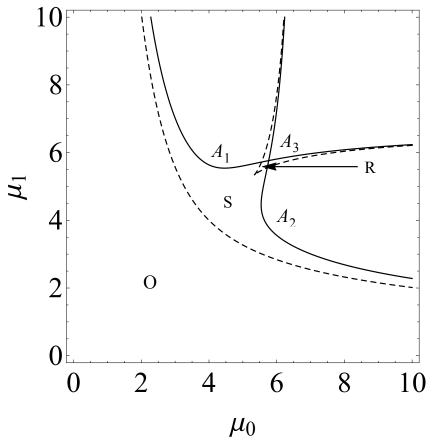

Fig. 8. Region of local stability, in the parameter space $\mu_0 O \mu_1$ where the fixed points of $f_1 \circ f_0$ are locally asymptotically stable and the maps are given by $f_i(x) = \mu_i x^2(1-x)$, $i = 0, 2$.

Let us start with the case $k = 2$. Following the techniques employed in [17], one can find the region of local stability of the fixed points of the composition map $\Phi_2 = f_1 \circ f_0$ by calculating the boundary where the absolute value of $\Phi_2'(x^*)$ is equal to one. This happens when

$$\begin{cases} f_1(f_0(x^*)) = x^* \\ f_1'(f_0(x^*))f_0'(x^*) = 1 \end{cases} \quad (13)$$

and

$$\begin{cases} f_1(f_0(x^*)) = x^* \\ f_1'(f_0(x^*))f_0'(x^*) = -1 \end{cases}. \quad (14)$$

Since the computations are long we will omit it here. Now, drawing implicitly, in the parameter space, the curves where the two previous equations are satisfied, we find the region where the stability of the fixed points of Φ_2 occurs. The stability regions are depicted, in the parameter space $\mu_0 O \mu_1$, in Fig. 8.

If the parameters μ_0 and μ_1 belong to the region O, then the origin is a fixed point globally asymptotically stable. Once the parameters cross the dashed curve, from Region O to Region S, a bifurcation occurs, known as saddle-node bifurcation. The fixed point $x^* = 0$ becomes unstable and a new locally stable fixed point of Φ_2 is born. This fixed point is, in fact, a $2-$periodic cycle of the $2-$periodic equation (2). Now if the parameters μ_0 and μ_1 cross the dashed curve from Region S to Region R, a saddle-node bifurcation occurs. The $2-$periodic cycle becomes unstable and a new locally asymptotically stable $2-$periodic cycle is born.

For a general framework of bifurcation in one-dimensional periodic difference equations, we refer the work of Elaydi, Luís, and Oliveira in [12].

Now, following the same techniques as before, we are able to find the regions of local stability of fixed points when $k = 3$ and $k = 4$. These regions are represented in Fig. 9. As we can observe, they are similar to the case $k = 2$ and the conclusions follow in the same fashion.

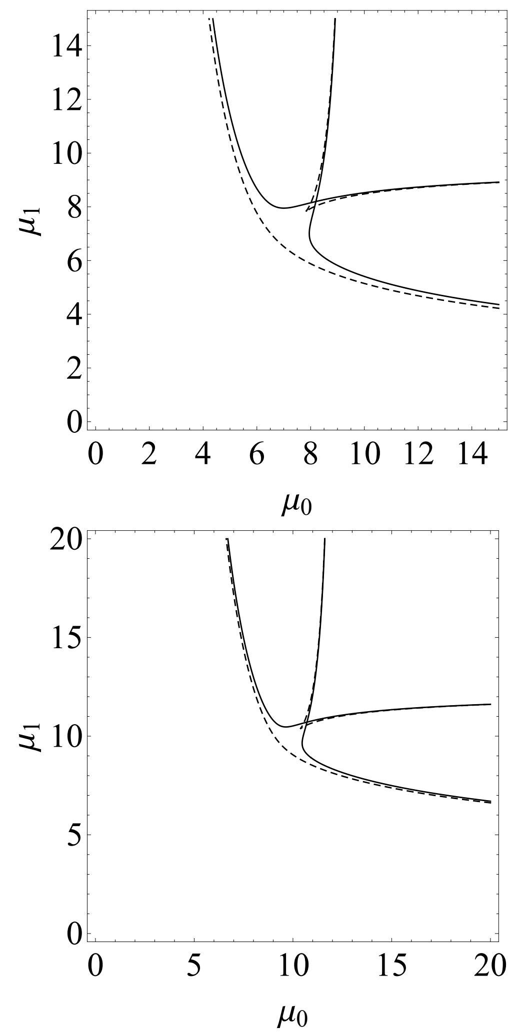

Fig. 9. Regions of local stability, in the parameter space, of the $2-$periodic equation when $k = 3$ (left) and $k = 4$ (right).

V. Conclusion

In this paper, we have presented a survey in local stability of discrete-time dynamical systems. The most important results concerning stability of hyperbolic and non-hyperbolic fixed points are addressed. Examples in both, autonomous and non-autonomous periodic models, are deeply studied. Some of these examples are widely used in the literature such as the Beverton-Holt model, the logistic model and the Ricker model. However, the examples with Allee effect are not so well known and studied. We should mention that, in the past two decades, the Allee effect was deeply studied in discrete dynamical systems.

Finally, this survey aims to be as a pedagogical instrument in stability analysis of discrete dynamical systems.

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